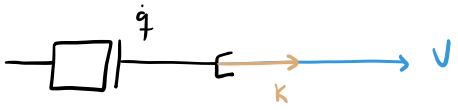
Velocity kinematics part 2

Velocity Kinematics:

- The velocity of the tool tip relative to the base.
- In order to know this, we need to know the velocities of each joint separately. So, at what linear velocities do the prismatic joints translate and at what angular velocities do the rotational joints rotate?
- We need velocity kinematics because when we want to control the motors of the robot, it isn't enough to only use the positions of the joints, but we need their velocities as well.

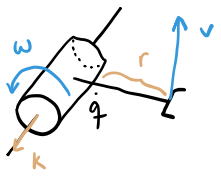
The velocity of a joint:

→ prismatic joint:



$V = \dot{q}K \qquad \omega = 0$

→ revolute joint:



$V = \dot{q}Kr \qquad \omega = \dot{q}K$

→ V is the linear velocity
→ ω is the angular velocity
→ $\dot{q}$ is the joint variables derivative - either θ or d
→ K is the unit vector

PS: We don't actually use this ↑ in calculations, but it's useful to see what values we are dealing with.

The Jacobian

Our end goal in velocity kinematics is to decide the velocity of the tool tip. This is given by ξ and the formula is:

$$\boxed{\xi = J\dot{q}}$$

→ J is the Jacobian: the transformation from joint variables to tool-tip velocity.
- a $6 \times n$ matrix where n = nr of joint variables

$V = J_v \dot{q}$
$\omega = J_w \dot{q}$
} joint velocity based on J

→ $\dot{q}$ is a vector with the velocity of each joint variable.

Finding the Jacobian

The Jacobian is built like this:

$$J = \begin{bmatrix} J_v \\ J_w \end{bmatrix}$$

→ J_v and J_w are $3 \times n$ matrices, where n are the number of joint variables.

→ J_v is the linear velocity part and J_w is the angular velocity part.

To fill in the Jacobian, we find J_v and J_w separately. Each column i in J_v and J_w is decided according to whether the i-th joint is revolute or prismatic. The formulas are:

$$J_{v_i} = \begin{cases} z_{i-1} \times (O_n - O_{i-1}) & \text{revolute} \\ z_{i-1} & \text{prismatic} \end{cases} \qquad J_{w_i} = \begin{cases} z_{i-1} & \text{revolute} \\ 0 & \text{prismatic} \end{cases}$$

But, where do these values come from? From the forward kinematics!! For example, if you have a robot with three joints, you will have three transformation matrices between the joints:

$$T_1^0 = \begin{bmatrix} a_1 & d_1 & g_1 & j_1 \\ b_1 & e_1 & h_1 & K_1 \\ c_1 & f_1 & i_1 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_3^0 = \begin{bmatrix} a_3 & d_3 & g_3 & j_3 \\ b_3 & e_3 & h_3 & K_3 \\ c_3 & f_3 & i_3 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^0 = \begin{bmatrix} a_2 & d_2 & g_2 & j_2 \\ b_2 & e_2 & h_2 & K_2 \\ c_2 & f_2 & i_2 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ the z's come from the third column

→ $z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ → always

→ z_1 comes from T_1^0 → $z_1 = \begin{bmatrix} g_1 \\ h_1 \\ i_1 \end{bmatrix}$

→ z_2 comes from T_2^0 → $z_2 = \begin{bmatrix} g_2 \\ h_2 \\ i_2 \end{bmatrix}$

→ the O's come from the fourth column

→ O_n is the origin of T_3^0 (this is O_3 as $n = 3$ joints)

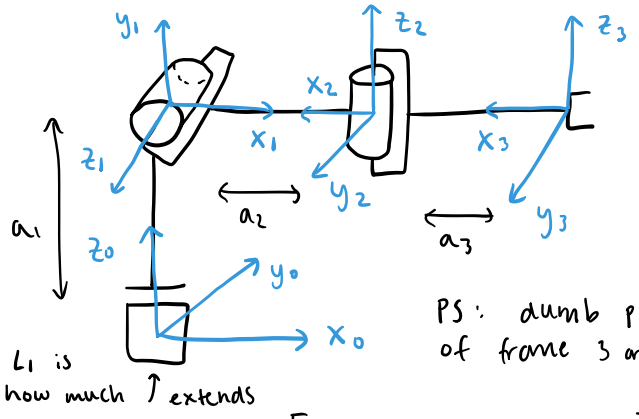
→ $O_0 = [0, 0, 0]$ → always

→ O_1 and O_2 come from T_1^0 and T_2^0 respectively

Now that we know what components the Jacobian is made out of, we can look at an example.

Velocity kinematics of an PRR manipulator

1) Find T_1^0, T_2^0 and T_3^0



L_1 is how much ↑ extends

PS: dumb placement of frame 3 and 4! :(

	θ_i	d_i	α_i	a_i
1	0	a_1^*	90°	0
2	$\theta_2^* + 180°$	0	90°	$-a_2$
3	θ_3^*	0	0	$-a_3$

$a_1^* = a_1 + L_1$
(link fixed length + extension)

$$T_1^0 = A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^0 = A_1 A_2 = H_1^0 H_2^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -c_2 & 0 & -s_2 & a_2 c_2 \\ -s_2 & 0 & c_2 & a_2 s_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -c_2 & 0 & -s_2 & a_2 c_2 \\ 0 & -1 & 0 & 0 \\ -s_2 & 0 & c_2 & a_2 s_2 + a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = A_1 A_2 A_3 = H_1^0 H_2^1 H_3^2 = \begin{bmatrix} -c_2 & 0 & -s_2 & a_2 c_2 \\ 0 & -1 & 0 & 0 \\ -s_2 & 0 & c_2 & a_2 s_2 + a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_3 & -s_3 & 0 & -a_3 c_3 \\ s_3 & c_3 & 0 & -a_3 s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -c_2 c_3 & c_2 s_3 & -s_2 & a_3 c_2 c_3 + a_2 c_2 \\ -s_3 & -c_3 & 0 & a_3 s_3 \\ -s_2 c_3 & s_2 s_3 & c_2 & a_3 s_2 c_3 + a_2 s_2 + a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2) Write the Jacobian, only using the components

$$J = \begin{bmatrix} z_0 & z_1 \times (O_3 - O_1) & z_2 \times (O_3 - O_2) \\ 0 & z_1 & z_2 \end{bmatrix} \begin{matrix} \rightarrow J_v \\ \rightarrow J_w \end{matrix}$$

prismatic joint → $i = 1$ | revolute joint → $i = 2$ | revolute joint → $i = 3$

3) Find all the components of the jacobian and write them

The z-vectors:

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = J_{v1}, \quad z_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = J_{w2}, \quad z_2 = \begin{bmatrix} -s_2 \\ 0 \\ c_2 \end{bmatrix} = J_{w3} \qquad (J_{w1} = \vec{0})$$

The O-vectors:

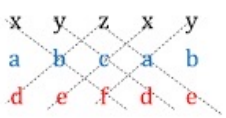
$$O_3 = \begin{bmatrix} a_3 c_2 c_3 + a_2 c_2 \\ a_3 s_3 \\ a_3 s_2 c_3 + a_2 s_2 + a_1 \end{bmatrix}$$

$$O_1 = \begin{bmatrix} 0 \\ 0 \\ a_1 \end{bmatrix} \qquad O_2 = \begin{bmatrix} a_2 c_2 \\ 0 \\ a_2 s_2 + a_1 \end{bmatrix}$$

$$O_3 - O_1 = \begin{bmatrix} a_3 c_2 c_3 + a_2 c_2 \\ a_3 s_3 \\ a_3 s_2 c_3 + a_2 s_2 \end{bmatrix} \qquad O_3 - O_2 = \begin{bmatrix} a_3 c_2 c_3 \\ a_3 s_3 \\ a_3 s_2 c_3 \end{bmatrix}$$

Calculating $z_1 \times (O_3 - O_1)$ and $z_2 \times (O_3 - O_2)$

cross product → $\vec{A} = [a, b, c]$ $\vec{B} = [d, e, f]$



$$\vec{A} \times \vec{B} = [(bf - ce), (cd - af), (ae - bd)]$$

x	y	z	x	y	
0	-1	0	0	-1	← z_1
$a_3c_2c_3 + a_2c_2$	a_3s_3	$a_3s_2c_3 + a_2s_2$	$a_3c_2c_3 + a_2c_2$	a_3s_3	← $o_3 - o_1$

$$z_1 \times (o_3 - o_1) = \begin{bmatrix} -a_3s_2c_3 - a_2s_2 \\ 0 \\ a_3c_2c_3 + a_2c_2 \end{bmatrix} = Jv_2$$

x	y	z	x	y	
$-s_2$	0	c_2	$-s_2$	0	← z_2
$a_3c_2c_3$	a_3s_3	$a_3s_2c_3$	$a_3c_2c_3$	a_3s_3	← $o_3 - o_2$

$$z_2 \times (o_3 - o_2) = \begin{bmatrix} a_3c_2s_3 \\ a_3c_3 \\ -a_3s_2s_3 \end{bmatrix} = Jv_3$$

4) Fill in the Jacobi matrix with these components

$$Jv = \begin{bmatrix} Jv_1 & Jv_2 & Jv_3 \\ z_0 & z_1 \times (o_3 - o_1) & z_2 \times (o_3 - o) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -a_3s_2c_3 - a_2s_2 & a_3c_2s_3 \\ 0 & 0 & a_3c_3 \\ 1 & a_3c_2c_3 + a_2c_2 & -a_3s_2s_3 \end{bmatrix}$$

$$Jw = \begin{bmatrix} Jw_1 & Jw_2 & Jw_3 \\ 0 & z_1 & z_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & s_2 \\ 0 & -1 & 0 \\ 0 & 0 & -c_2 \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & -a_3s_2c_3 - a_2s_2 & a_3c_2s_3 \\ 0 & 0 & a_3c_3 \\ 1 & a_3c_2c_3 + a_2c_2 & -a_3s_2s_3 \\ 0 & 0 & s_2 \\ 0 & -1 & 0 \\ 0 & 0 & -c_2 \end{bmatrix} \rightarrow$$ Velocity kinematics of the PRR robot

Using the Jacobian

Now that we have our Jacobian, we can use it to do some calculations. For example, we can use the jacobian to calculate the velocity of the tool tip given the velocity of the joint variables.

Velocity of joint variables: $\dot{q} = \begin{bmatrix} \dot{d}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} 0.0003 \text{ m/s} \\ 0.7 \text{ rad/s} \\ 0.02 \text{ rad/s} \end{bmatrix}$

The joint variables are given as $a_1 = 321$ mm, $\theta_2 = 30°$, $\theta_3 = 60°$, $a_2 = 100$ mm, $a_3 = 207$ mm

random values chosen for this task

What is the velocity of the tool tip?

$\xi = J\dot{q}$ → the general formula

Usually, we don't want both the linear and angular velocities, but just the linear velocity. Therefore we can split ξ like this:

$$\xi = \begin{bmatrix} v \\ w \end{bmatrix}$$

Here we have

$v = J_v \dot{q}$ and $w = J_w \dot{q}$

$$\begin{matrix} a_3 s_2 c_3 + a_2 s_2 & a_3 c_2 s_3 \\ 0 & -a_3 c_3 \\ -a_3 c_2 c_3 - a_2 c_2 & a_3 s_2 s_3 \end{matrix}$$

So since we only want the linear velocity of the tool tip, we use $v = J_v \dot{q}$:

$$v = J_v \dot{q}$$

$$= \begin{bmatrix} 0 & -a_3 s_2 c_3 - a_2 s_2 & a_3 c_2 s_3 \\ 0 & 0 & a_3 c_3 \\ 1 & a_3 c_2 c_3 + a_2 c_2 & -a_3 s_2 s_3 \end{bmatrix} \begin{bmatrix} 3 \text{ mm/s} \\ 0.7 \text{ rad/s} \\ 0.02 \text{ rad/s} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -101,75 & 155,25 \\ 0 & 0 & 103,50 \\ 1 & 176,24 & -89,633 \end{bmatrix} \begin{bmatrix} 3 \text{ mm/s} \\ 0.7 \text{ rad/s} \\ 0.02 \text{ rad/s} \end{bmatrix} = \begin{bmatrix} -68.12 \\ 2.07 \\ 124,58 \end{bmatrix}$$

So this is the velocity of the tool tip relative to the base frame in the x, y and z directions.

singularities

- Configuration of the robot where motion in some direction is impossible. The tooltip becomes blocked in that position.
- Minor movement of the tool could require "infinite" force in the joints - so a minor movement in theory could require an "infinite" force to actually do this movement.
- "Doesn't mean that joints will fly off into the stratosphere at superluminal speeds, but the joints might strain and break".
  — an old group teacher

  → this means that if you don't take the singularities into account, the joints might end up in singular configurations and they can be destroyed.
- Mathematically, singularities happen when rank $J(q) < max$. The maximum value is the number of columns in the Jacobian and rank $J(q)$ is the number of linearly independent columns in $J(q)$. This means that if rank $J(q) < max$ for a set of joint variables q, then the manipulator is in a singular configuration.
- We can also find singularities using the jacobian. This is given by the determinant of J_v. So in order to find the singularities of a robot we would solve this:

$$|J_v| = 0 \qquad (|J| = 0)$$

- We can also find singularities by just observing the robot and thinking about where it could go wrong.
- Singularities may correspond to points on the boundary of the workspace - these are points of maximum reach of the manipulator.
- Near singularities, a unique solution to the inverse kinematics doesn't exist. In such cases there may be no solution or there may be infinitely many solutions

Singularity of the PRR (ish) manipulator

→ we do this analytically, by solving $|J_v| = 0$.

The determinant of a 3×3 matrix is given by

$$J = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

PS: you can either do this by hand or with python or matlab :).

$$\det(J) = \det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right)$$

$$\det(J) = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - \mathrm{b} \begin{vmatrix} d & f \\ g & i \end{vmatrix} + \mathrm{c} \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\det(J) = a\,(e * i - f * h) - \mathrm{b}(d * i - f * g) + \mathrm{c}(\mathrm{d} * \mathrm{h} - \mathrm{e} * \mathrm{g})$$

$$\det(J) = a\,(e * i - f * h) - \mathrm{b}(d * i - f * g) + \mathrm{c}(\mathrm{d} * \mathrm{h} - \mathrm{e} * \mathrm{g}) = 0$$

Since we can only find the determinant of square matrices, we cannot use the whole Jacobian. Therefore, we decouple the J_v component from J by itself.

$$J_v = \begin{bmatrix} 0 & -a_3 s_2 c_3 - a_2 s_2 & a_3 c_2 s_3 \\ 0 & 0 & a_3 c_3 \\ 1 & a_3 c_2 c_3 + a_2 c_2 & -a_3 s_2 s_3 \end{bmatrix}$$

Finding the singularities by $|J_v|$:

$$\det(J_v) = \det\left(\begin{bmatrix} 0 & -a_3 s_2 c_3 - a_2 s_2 & a_3 c_2 s_3 \\ 0 & 0 & a_3 c_3 \\ 1 & a_3 c_2 c_3 + a_2 c_2 & -a_3 s_2 s_3 \end{bmatrix}\right)$$

$$= \det\begin{pmatrix} -a_3 s_2 c_3 - a_2 s_2 & a_3 c_2 s_3 \\ 0 & a_3 c_3 \end{pmatrix}$$

$$= (-a_3 s_2 c_3 - a_2 s_2)(a_3 c_3)$$

$$= -a_3^2 s_2 c_3^2 - a_2 a_3 s_2 c_3$$

Setting $\det(J_v)$ equal to 0:

$$-a_3^2 s_2 c_3^2 - a_2 a_3 s_2 c_3 = 0 \quad \Rightarrow \quad a_3^2 s_2 c_3^2 + a_2 a_3 s_2 c_3 = 0$$

$$a_3 s_2 c_3 (a_3 c_3 + a_2) = 0$$

Either $a_3 s_2 c_3 = 0$ or $a_3 c_3 + a_2 = 0$

a_3 will never be 0, so $s_2 c_3 = 0$.

$$\Rightarrow \sin(\theta_2)\cos(\theta_3) = 0$$

This happens when either θ_2 is 0 or 180° or when θ_3 is 90° or −90°.

$$c_3 = -\frac{a_2}{a_3}$$

Now we have found the joint variables that lead to singular configurations for this robot.