CONTROL THEORY part 2

Dynamic model of the robot:

$$J\ddot{q} + B\dot{q} + D = f$$

Laplace transform: time domain → frequency domain
(the point is to get linear equations instead of differential equations)

→ we do the laplace transform using these 2 rules:

1) differentiation in time domain becomes multiplication by s in laplace domain

2) integration in time domain becomes division by s in laplace domain

For example: $J\ddot{q} + B\dot{q} + D = f \rightarrow Js^2\theta + Bs\theta + D = f$

Setpoint controllers: controllers that drive a robot to a given setpoint. These controlles in principle work by measuring the variable (theta - the angle of the actuator) and comparing it to the setpoint The controller then uses a feedback loop to adjust the actuator until the variable reaches and maintains the desired set point.

$$e = \theta_d - \theta$$
$$\dot{e} = \dot{\theta}_d - \dot{\theta} = -\dot{\theta}$$

The types of controllers:

- Proportional (P): $U(t) = K_p e(t)$ (time)
  $U(s) = K_p E(s)$ (laplace)
- Proportional derivative (PD): $U(t) = K_p e(t) + K_d \dot{e}(t)$
  $U(s) = K_p(\theta_d - \theta) - K_d s\theta$
- PID: $U(t) = K_p e(t) + K_d \dot{e}(t) + K_i \int e(t)\,dt$
  $U(s) = (K_p + K_d s + K_i/s)\,E(s)$

Transfer functions:

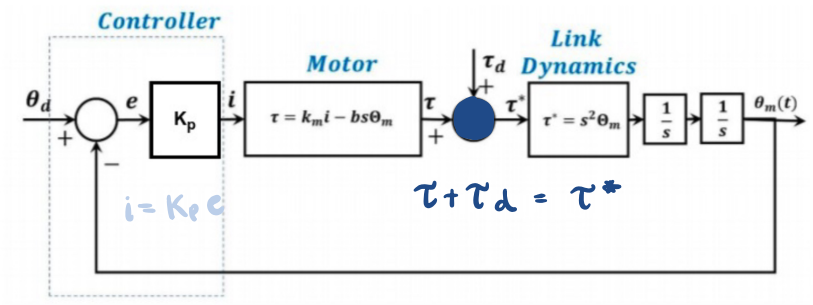
$$H(s) = \frac{Y(s)}{X(s)}$$ → output / → input

↳ If we set the denominator to 0, we can solve the equation to find the zeros of the equation. This is useful for determining the stability of the system.

the characteristic polynomial $\Omega(s)$

→ If your task is to find the transfer function the easiest approach is to set the controller equal to the dynamic system equation and solve for θ.

Example 2021 exam:



Task: find the transfer function between input θ_d and θ_m. τ_d is 0 and a P controller is used.

→ sometimes there are different ways to read the diagrams, which will lead to different methods of solving and different results

Method 1: find $U_{control}$ (controller equation) and U_{sys} (dynamic equation for system) and set them equal to each other by reading the diagram.

$U_{control} = K_p(\theta_m - \theta_d)$

$U_{sys} = k_m i - bs\theta_m + s^2\theta_m$

} $U_{control} = U_{sys}$ and solve for θ_m

Method 2: look at how the block diagram is structured and get an equation from that.

In this case we can see a summation sign that combines the left and right part of the diagram according to this equation:

$$\tau^* = \tau + \tau_d$$

1) We know that $\tau_d = 0$
2) We have τ from the left side of the block diagram: $\tau = K_m i - b s \theta_m$.
3) From the block diagram we also have that $i = K_p e(s) = K_p(\theta_d - \theta_m)$.
4) We insert this into τ: $K_p K_m (\theta_d - \theta_m) - b s \theta_m$
5) We have τ^* from the left side: $\tau^* = s^2 \theta_m$
6) We assemble the equation:

$$\tau = \tau^* \Rightarrow K_p K_m (\theta_d - \theta_m) - b s \theta_m = s^2 \theta_m$$

7) Then we solve for θ_m:

$$K_p K_m \theta_d - K_p K_m \theta_m - b s \theta_m = s^2 \theta_m$$

$$K_p K_m \theta_d = \theta_m (s^2 + K_p K_m + b s)$$

$$\theta_m = \frac{K_p K_m \theta_d}{s^2 + b s + K_p K_m}$$ → the transfer function

Estimation of controller constants
(meaning K_p, K_d and K_i)

→ this is determined by the characteristic polynomial ($\Omega(s)$) from the transfer function (the determinant)

We use a general damped second order system. This is given by

$$s^2 + 2\zeta \omega s + \omega^2 = 0$$

ζ: damping ratio
→ if $\zeta = 1$, the system is critically damped

ω: closed loop natural frequency

→ A general damped second order system is a mathematical model used to describe the behavior of a system that can oscillate around an equilibrium point in response to an external force.

→ In robotics we usually use a critically damped system (where $\zeta = 1$). This is a control system that is designed to return to its equilibrium state as quicly as possible without overshooting.

→ When a robotic system is subjected to external forces, it can oscillate around its equilibrium position. If the damping is too low, the system will oscillate with a high amplitude and will take a long time to return to the equilibrium. If the damping is too high, the system will return to equilibrium without overshooting but will take a longer time.

A critically damped system strikes a balance between these two extremes by damping the oscillations just enough to prevent overshoot, while still returning to the equilibrium position as quickly as possible.

The critically damped system will have equation:

$$s^2 + 2ws + w^2 = 0$$

So, to find formulas for the control constants, we use the idea of a critically damped system with the determinant (the characteristic polynomial) of the transfer function.

What we do is to get the characteristic polynomial on the form of the generally damped system (sometimes it is already on that form) and set it to 0. Then match the terms in both expressions with each other.

Example:

$\Omega(s) = Js^2 + Bs + K_P$ (this is the characteristic polynomial of the transfer function you found last session!)

$\Omega(s) = 0 \rightarrow Js^2 + Bs + K_P = 0$

1) set $\Omega(s)$ to 0

$$s^2 + \frac{B}{J}s + \frac{K_P}{J} = 0$$

2) get $\Omega(s)$ to the form of a general / critical system.

the task will tell you what kind of system we are dealing with - like if it's critically damped etc.

We know that a critical system and our system is given by:

$s^2 + 2\omega s + \omega^2 = 0$ and

$s^2 + \frac{B}{J}s + \frac{K_P}{J} = 0$

3) Then we can do some matching between these 2 to obtain a formula for the control constant K_P.

$$\omega^2 = \frac{K_P}{J} \Rightarrow \boxed{K_P = J\omega^2}$$

Finally, sometimes we want an expression for the steady state error.

→ The steady state error is the difference between the desired output of a control system and the actual output when the system has reached a steady state / a stable operating condition.

→ It is a measure of accuracy of the control system in tracking the reference input over time, after any transient response has died out.

To find the steady state error, we use the final value theorem. This relates the frequency domain to the time domain as time approaches infinity. This theorem says:

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sf(s)$$

The steady state error is the error when $t \to \infty$:

$$e_{ss} = \lim_{s \to 0} sE(s)$$

So, this is the formula we would use to find the steady state error. This means that we have to find an expression for the error $(\theta d - \theta)$ using an expression for θ and solve this.

Here is an example:

Again we use the example from last session, where the transfer function was:

$$\theta = \frac{K_p \theta d - D}{Js^2 + Bs + K_p} = \frac{K_p \theta d - D}{\Omega} \qquad \Omega = Js^2 + Bs + K_p$$

We then define the error and rewrite it a bit:

$$E(s) = \theta d - \theta = \theta d - \frac{K_p \theta d - D}{Js^2 + Bs + K_p}$$

$$= \theta d - \frac{K_p \theta d}{Js^2 + Bs + K_p} + \frac{D}{Js^2 + Bs + K_p}$$

$$= \frac{\theta d (Js^2 + Bs + K_p) - K_p \theta d}{Js^2 + Bs + K_p} + \frac{D}{Js^2 + Bs + K_p}$$

$$= \frac{Js^2 \theta d + Bs \theta d + K_p \theta d - K_p \theta d}{Js^2 + Bs + K_p} + \frac{D}{Js^2 + Bs + K_p}$$

$$= \frac{\theta d (Js^2 + Bs)}{Js^2 + Bs + K_p} + \frac{D}{Js^2 + Bs + K_p}$$

Then, to evaluate the response of the system, we use a step reference input. This is an input signal type that consists of a sudden change/step in reference value at a specific time, followed by a constant value after that time. Then this is used to evaluate how the system responds to such a sudden change.

To do this, we need to make some adjustments and therefore we set

$$\theta_d(s) = \frac{\Omega^d}{s} \quad \text{and} \quad D(s) = \frac{D}{s}$$

$$\Omega = Js^2 + Bs + \omega^2$$

→ division by s allows us to apply the final value theorem to directly calculate the steady state theorem

Then we can use the formula:

$$\lim_{s \to 0} sE(s) = \frac{s \cdot \frac{\Omega^d}{s}(Js^2 + Bs)}{\Omega} + \frac{s \cdot \frac{D}{s}}{\Omega}$$

$$= \frac{\Omega^d(Js^2 + Bs)}{\Omega} + \frac{D}{\Omega}$$

$$= \frac{\Omega^d(Js^2 + Bs)}{Js^2 + Bs + Kp} + \frac{D}{Js^2 + Bs + Kp}$$

$Js^2 + Bs = 0$ when $s \to 0$; $Kp \neq 0$ when $s \to 0$

If $s \to 0$ then we have:

$$e_{ss} = \frac{D}{Kp}$$

→ this is the formula for the steady state error