Dynamics part 1 - Lagrangian

Dynamics: the study of a robot's motion and the forces and torques that cause the motion. It involves for example applications from concepts from physics to model the behavior of the robot. This is important for controlling robotic systems and for understanding how the robot interacts with the environment. This can also be used to design control algorithms to adjust the forces and torques applied to the robot in order to move it.

When dealing with a manipulator and we want it to move it from its starting position to a target position, we need to know the dynamic properties of the manipulator so that we can predict how the various forces and accelerations will affect the movement.

- If for example too small forces are applied to produce movement we may have a too slow movement.
- If the forces are too strong, we may risk making the manipulator collide with other objects in the environment or have it swinging excessively with respect to the desired position.

This is why we need the robot's dynamics.

Dynamics can be divided into two areas: **kinematics** (which we already have looked at) and **kinetics**. Kinematics deals with the description of motion without considering the forces and torques that cause the motion. Kinetics takes the forces and torques into consideration.

So, let's start with looking at the dynamics of a robot.

A typical task in dynamics is: derive the dynamical model for the manipulator using the Euler-Lagrange equations

Euler-Lagrange:

torque-expression for joint k

derivative with respect to time

Lagrangian

what is the lagrangian? we need to find it!

$$\tau_k = \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}$$

derivative of joint variables for joint k

The full robot dynamic model can be written as a vector containing τ_k for each joint $k = 1, \dots, n$.

$$\tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix}$$

- n is the number of joints
- $\tau_1 - \tau_n$ are the Euler-Lagrange expressions for each joint.

The dynamic model is represented then by the vector τ of joint torques that act on the robot's joints.

The dynamic model includes equations that describe the relationship between the joint torques, the robot's current joint positions and velocities and it's and other physical properties.

The torques for each joint represented by τ can be used to control the motion of the robot. So by adjusting the torques you can gain precise control of the robot.

- The dynamic model can also be rewritten:

$$\tau = D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q)$$

torque to be applied to achieve desired motion → τ calculated based on the current state of the robot (positions, velocities and acceleration.

- The goal is usually to express the dynamics on this form

General steps to find the dynamics:

1) Find the Lagrangian term L.
   → entails finding kinetic and potential energy of each mass/body

2) Use the Euler-Lagrange equation to find τ_K for each joint
   → We use the Lagrangian L for this.
   → We have to differentiate :)

3) Rewrite the dynamic model into this form (if required in task)
   $\tau = D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q)$

Components of the dynamic model for a manipulator

$$\tau = D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q)$$

This can be written as following for a 3DOF (3joint) manipulator:

$$\tau = \underbrace{\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}}_{D}\ddot{q} + \underbrace{\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}}_{C}\dot{q} + \underbrace{\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}}_{g}$$

Each term in the dynamical model represents forces that act on the robot (this is described in detail in a book called "Robotics" by Bruno Siciliano (you can find it if you search for it on google ü)).

Torque: the force that causes rotational motion. When a torque is applied to an object, it causes it to rotate around an axis.

Inertia / treghet: a property of objects that says that they will remain at rest or in unchanging motion unless an external force acts on the robot.

- $D(q)\ddot{q}$ → the acceleration term
  - $D(q)$ describes the inertia
    → accounts for how the mass is distributed in the robot and how that affects the acceleration of the robot.
    - The diagonal terms of the matrix $D(q)$ represents the moment of inertia at joint i axis in the manipulator while the other joints are blocked. It's saying something about how heavy each joint is by itself and how hard it is to move by itself.
    - All other terms represent the effect on the joints on each other. This tells us how hard it is to move a joint because of how it's connected to other parts.

- $C(q, \dot{q})\dot{q}$ → the quadratic velocity term
  - $C(q, \dot{q})$ describes the centrifugal and coriolis forces
    → Describes the forces that arise because of the robots own motion and the interaction between different body parts as the robot moves.
    - The diagonal terms describe the centrifugal effect.
      → the force that makes you feel like you're being pushed outwards (ex when driving too fast in a curb.
    - All other terms describe the coriolis effect.
      → what makes a straight path curved when you are moving on something that's already moving.

- $g(q)$ → the gravity term
  - A vector that describes the force exerted on the robot due to gravity.
  → important to understand how much force needs to be applied to counteract gravity.

Btw, what are $\ddot{q}$, $\dot{q}$ and q?

- $\ddot{q}$ and $\dot{q}$ and q are vectors
  - q is the **joint position vector**. It represents the robot's joints positions

$$q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

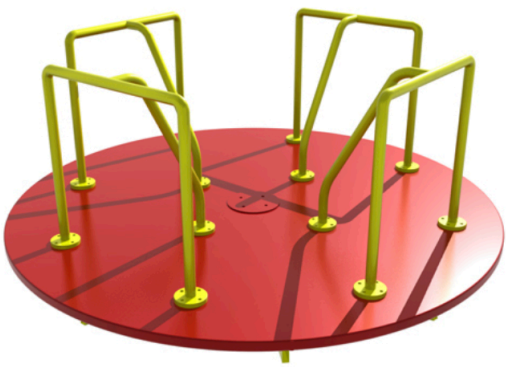
  - $\dot{q}$ is the velocity of the joint variable
    - either $\dot{\theta}$ or $\dot{d}$ based on joint type
    - **joint velocity vector**

$$\dot{q} = \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

  - $\ddot{q}$ is the acceleration of the joint variable
    - either $\ddot{\theta}$ or $\ddot{d}$ based on joint type
    - **joint acceleration vector**

$$\ddot{q} = \begin{bmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_n \end{bmatrix}$$

Simple example:



→ The merry-go-around is heavy so it needs more force to move or stop which is what the **D(q)** matrix describes.

→ When the merry-go-around is spinning and you hold on, you feel a force pulling you outwards which is the centrifugal force.

→ If you try to walk on the merry-go-around in a straight line, it will be very hard and that's the coriolis effect. These two are described by **C(q,q̇)** matrix.

Lagrangian intro

- Before we get to the dynamical model of the robot, we have to do some calculations.
- The first step is to calculate the Lagrangian term.

Lagrangian: The difference between kinetic energy and potential energy.

- $L = K - P$
- We have to make some assumptions about our manipulator in order to be able to calculate the energies.

these are usually indicated in the task more or less clearly :)

↓

no assumptions will make calculations HARD

- **Body composition**: a joint and the attached link can be counted as one body/mass or not - some links may count as massless. The task usually presents how many masses there are and what they represent.
- **Body shape**: a link with or without its joint can be considered a rectangular / cylindrical solid.
- **Mass distribution**: mass can be distributed uniformly (or non-uniformly) or concentrated in a single point (point mass).

These assumptions influence how we calculate the dynamical model. Assumptions about body compositions dictate how many masses are present and assumptions about body shape and mass distribution determine how we calculate the energy. ⇒ Read the task carefully.

Body composition: how many masses are present and how do the bodies look? Are we considering joints + links as one mass / body? Or are we considering joints and links separately? A mix?

Mass distribution: uniformly distributed or point mass?

→ point mass: mass is concentrated at one point and that's where the centre of mass is too. Inertia tensor is always 0.

→ uniformly distributed: mass is evenly distributed over the mass / body and the center of mass is in the middle of the body / mass.

Body shape: will affect the inertia matrix I_i, but this matrix is usually given to us.

Finding the Lagrangian - components

$L = K - P$ = kinetic energy - potential energy.

- **Kinetic energy**:

$$K = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T \mathcal{I} \omega$$ For one-link

$$K = \frac{1}{2} \dot{q}^T \left[\sum_{i=1}^{n} m_i J_{vi}(q)^T J_{vi}(q) + J_{\omega i}(q)^T R_i(q) I_i R_i(q)^T J_{\omega i}(q) \right] \dot{q}$$ For n-DOF links

- $v_i = J_v(q)\,\dot{q}$ - $\omega_i = J_\omega(q)\,\dot{q}$
- $\mathcal{I}_i$ - inertia tensor ($\mathcal{I}_i = R_i I_i R_i^T$ in general)
  → inertia tensor of mass m_i with respect to the base frame.

also called inertia tensor of mass m_i - but it's with respect to the coordinate frame attached to body of mass m_i.

I_i = $\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$ - gives us an idea about how mass is distributed in a rigid body.

→ I_i is usually given and you have to find $\mathcal{I}_i$ yourself. This means that you have to find R_i too and multiply with I_i according with the formula above to get $\mathcal{I}_i$.

$I_i \neq \mathcal{I}_i$

- R_i - rotation matrices

$$R_i = \begin{bmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{bmatrix}$$

from base frame to the coordinate frame of the mass

can be calculated using body shape. Oscar will show you :)

So, what you do is assign a coordinate frame to the mass centre in question and find the rotation R between base and this coordinate system. Btw, how you put the frame on the centre of mass is up to you.

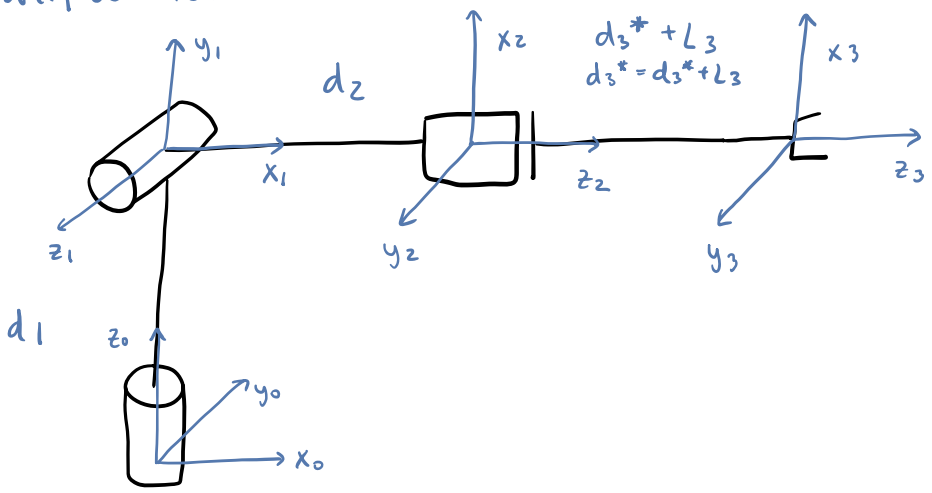
- **Potential energy**

$$P = mgh$$ For one-link

- $h = r_{ci}$ = height

$$P = \sum_{i=1}^{n} m_i g^T r_{ci}$$ For n-DOF links

Example - how to find the lagrangian for a spherical manipulator:



1) Find the Jacobian for the manipulator

$$T_1^0 = \begin{bmatrix} s_1 & 0 & s_1 & 0 \\ c_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Transformation matrices

$$T_2^0 = \begin{bmatrix} -c_1c_2 & s_1 & c_1c_2 & 0 \\ -s_1s_2 & -c_1 & c_2s_1 & 0 \\ c_2 & 0 & s_2 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = \begin{bmatrix} -c_1c_2 & s_1 & c_1c_2 & c_1c_2(d_3+d_2) \\ -s_1s_2 & -c_1 & c_2s_1 & c_2s_1(d_3+d_2) \\ c_2 & 0 & s_2 & d_1 + s_2(d_3+d_2) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Helpful variables:

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad z_1 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}, \quad z_2 = \begin{bmatrix} c_1c_2 \\ c_2s_1 \\ s_2 \end{bmatrix}$$

$$O_3 - O_0 = \begin{bmatrix} c_1c_2(d_3+d_2) \\ c_2s_1(d_3+d_2) \\ d_1 + s_2(d_3+d_2) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1c_2(d_3+d_2) \\ c_2s_1(d_3+d_2) \\ d_1 + s_2(d_3+d_2) \end{bmatrix}$$

$$O_3 - O_1 = \begin{bmatrix} c_1c_2(d_3+d_2) \\ c_2s_1(d_3+d_2) \\ d_1 + s_2(d_3+d_2) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix} = \begin{bmatrix} c_1c_2(d_3+d_2) \\ c_2s_1(d_3+d_2) \\ s_2(d_3+d_2) \end{bmatrix}$$

Finding J_v and J_w:

$$J_{v_1} = z_0 \times (o_3 - o_0) = \begin{bmatrix} -c_2 s_1 (d_3 + d_2) \\ c_1 c_2 (d_3 + d_2) \\ 0 \end{bmatrix}$$

$$J_{v_2} = z_1 \times (o_3 - o_1) = \begin{bmatrix} -c_1 s_2 (d_3 + d_2) \\ -s_1 s_2 (d_3 + d_2) \\ c_2 (d_3 + d_2) \end{bmatrix}$$

$$J_{v_3} = z_2 = \begin{bmatrix} c_1 c_2 \\ c_2 s_1 \\ s_2 \end{bmatrix}$$

$$J_{w_1} = z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J_{w_2} = z_1 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}, \quad J_{w_3} = 0$$

$J_{v_i} = z_{i-1} \times (o_n - o_{i-1})$	rev.
$J_{v_i} = z_{i-1}$	lin.
$J_{w_i} = z_{i-1}$	rev.
$J_{w_i} = 0$	lin.

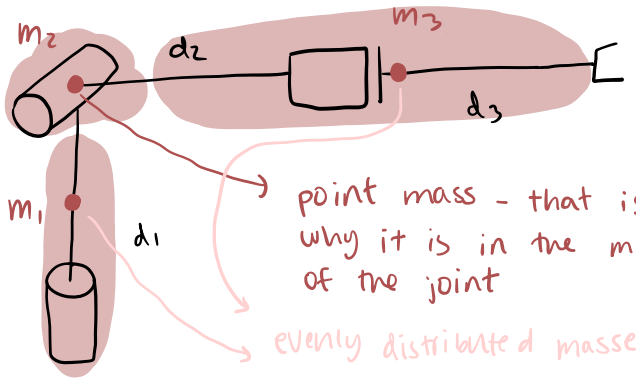
Putting it together yields the jacobian matrix

$$J = \begin{bmatrix} J_v \\ J_w \end{bmatrix} = \begin{bmatrix} J_{v_1} & J_{v_2} & J_{v_3} \\ J_{w_1} & J_{w_2} & J_{w_3} \end{bmatrix}$$

we need the jacobian because finding kinetic energy involves using velocities

$$= \begin{bmatrix} -c_2 s_1 (d_3 + d_2) & -c_2 s_1 (d_3 + d_2) & c_1 c_2 \\ c_1 c_2 (d_3 + d_2) & -s_1 s_2 (d_3 + d_2) & c_2 s_1 \\ 0 & c_2 (d_3 + d_2) & s_2 \\ 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{matrix} \} J_v \\ \\ \\ \} J_w \\ \\ \end{matrix}$$

Centre of masses:



Assumptions:

1) Mass is evenly distributed for masses 1 and 3, while 2 is a point mass
2) Body composition:
   - → Body 1: joint 1 with link 1 attached
   - → Body 2: joint 2 alone
   - → Body 3: joint 3 + attached links

2) Finding the Kinetic energy

- We find the kinetic energy for each mass separately. and add the kinetic energies for each mass to get the total kinetic energy.
  - since our example has 3 joints and thus 3 separate masses, the total kinetic energy is: $K = K_1 + K_2 + K_3$.
  - we calculate kinetic energy for each mass present and not the joints/links.
- We use the jacobian matrix as a starting point.
- The idea is that we calculate the kinetic energy of each mass separately and completely ignore the next masses.
  - So when calculating K_1 we only look at mass 1 and ignore all variables that deal with masses 2 and 3.
  - when calculating K_2 we look at masses 1 and 2 and ignore variables that deal with mass 3.
  - For mass 3, we look at the whole robot.

By pretending that only joints up to the i-th exists.

so, the jacobian matrix relates the velocity of the end effector to the velocities of the joints.

when calculating the kinetic energy for the first mass, the other columns are set to 0 in the jacobian because they represent the contribution of the remaining masses to the energy of the system. since we want only the kinetic energy of the first mass, we don't need to consider the effects of the other masses on the kinetic energy of the system. The same goes for the next mass and so on.

Finding K_1: $K_1 = \frac{1}{2} m_1 v_1^T v_1 + \frac{1}{2} \omega_1^T I_1 \omega_1$

→ we set all values that are connected to mass 2 and 3 to 0 (because we only focus on mass 1)

→ we set column 2 and 3 in the jacobian to 0 because they are related to mass 2 and 3.

This gives us:

$$J_v = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad J_\omega = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

→ we have inertia tensor $I_1 = 0$ (defined usually in the task). Be aware that I_1 is a matrix, so since it's 0, it will be filled with 0s.

→ The linear and angular velocities are:

we need the linear and angular velocity of each mass

$$v_1 = J_{v_1}\dot{q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\omega_1 = J_{\omega_1}\dot{q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix}$$

→ We put it all together to get K_1:

$$K_1 = \tfrac{1}{2} m_1 v_1^T v_1 + \tfrac{1}{2} \omega_1^T I_1 \omega_1$$

$$= \frac{1}{2} m_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix} = 0$$

Finding K_2: $K_2 = \tfrac{1}{2} m_2 v_2^T v_2 + \tfrac{1}{2} \omega_2^T I_2 \omega_2$

→ we set all values that are connected to mass 3 to 0 (because we only focus on masses 1 and 2).

→ we set column 3 in the jacobian to 0 because it is related to mass 3.

This gives us:

$$J_v = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad J_\omega = \begin{bmatrix} 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

→ we have inertia tensor $I_2 = 0$ here too

→ The linear and angular velocities are:

$$v_2 = J_{v_2}\dot{q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\omega_2 = J_{\omega_2}\dot{q} = \begin{bmatrix} 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} \dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix}$$

→ we put it all together to get K_2:

$$K_1 = K_2 = \tfrac{1}{2} m_2 v_2^T v_2 + \tfrac{1}{2} \omega_2^T I_2 \omega_2$$

$$= \frac{1}{2} m_2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} \dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix} = 0$$

Finding K_3: $K_3 = \tfrac{1}{2} m_3 v_3^T v_3 + \tfrac{1}{2} \omega_3^T I_3 \omega_3$

→ we don't set any values to 0 because we are calculating the kinetic energy for the last mass and now all previous masses are included.

This gives us:

NOTE: we also modify the jacobian so that it matches where centre of mass is located on the robot. So since m_3 is in the middle of $d_2 + d_3$, we must take this into account by dividing by 2 as that's where the centre of mass lies.

$$J_v = \begin{bmatrix} -\frac{1}{2} c_2 s_1 (d_3 + d_2) & -\frac{1}{2} c_2 s_1 (d_3 + d_2) & c_1 c_2 \\ \frac{1}{2} c_1 c_2 (d_3 + d_2) & -\frac{1}{2} s_1 s_2 (d_3 + d_2) & c_2 s_1 \\ 0 & \frac{1}{2} c_2 (d_3 + d_2) & s_2 \end{bmatrix}$$

$$J_w = \begin{bmatrix} 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

this is because when we calculate kinetic energy we focus on where the centre of masses are and no longer for each joint.

→ we have inertia tensor $I_3 = 0$ here too

→ the linear and angular velocities are:

$$v_3 = J_{v3} \dot{q} = \begin{bmatrix} -\frac{1}{2} c_2 s_1 (d_3 + d_2) & -\frac{1}{2} c_2 s_1 (d_3 + d_2) & c_1 c_2 \\ \frac{1}{2} c_1 c_2 (d_3 + d_2) & -\frac{1}{2} s_1 s_2 (d_3 + d_2) & c_2 s_1 \\ 0 & \frac{1}{2} c_2 (d_3 + d_2) & s_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} c_2 s_1 (d_3 + d_2) \dot{q}_1 - \frac{1}{2} c_2 s_1 (d_3 + d_2) \dot{q}_2 + c_1 c_2 \dot{q}_3 \\ \frac{1}{2} c_1 c_2 (d_3 + d_2) \dot{q}_1 - \frac{1}{2} s_1 s_2 (d_3 + d_2) \dot{q}_2 + c_2 s_1 \dot{q}_3 \\ \frac{1}{2} c_2 (d_3 + d_2) \dot{q}_2 + s_2 \dot{q}_3 \end{bmatrix}$$

$$\omega_3 = J_{w3} \dot{q} = \begin{bmatrix} 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} \dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix}$$

→ we put it all together to get K_3:

$$K_3 = \tfrac{1}{2} m_3 v_3^T v_3 + \tfrac{1}{2} \omega_3^T I_3 \omega_3$$

$$= \frac{1}{2} m_3 \begin{bmatrix} -c_2 s_1 (d_3+d_2)\dot{q}_1 - c_2 s_1 (d_3+d_2)\dot{q}_2 + c_1 c_2 \dot{q}_3 \\ c_1 c_2 (d_3+d_2)\dot{q}_1 - s_1 s_2 (d_3+d_2)\dot{q}_2 + c_2 s_1 \dot{q}_3 \\ c_2 (d_3+d_2)\dot{q}_2 + s_2 \dot{q}_3 \end{bmatrix}^T$$

$\frac{1}{2}$ should be here like above, so because of this there is a small following error in the rest of the calculations

$$\begin{bmatrix} -c_2 s_1 (d_3+d_2)\dot{q}_1 - c_2 s_1 (d_3+d_2)\dot{q}_2 + c_1 c_2 \dot{q}_3 \\ c_1 c_2 (d_3+d_2)\dot{q}_1 - s_1 s_2 (d_3+d_2)\dot{q}_2 + c_2 s_1 \dot{q}_3 \\ c_2 (d_3+d_2)\dot{q}_2 + s_2 \dot{q}_3 \end{bmatrix}$$

$$+ \frac{1}{2} \begin{bmatrix} \dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix}$$

$$= \frac{1}{2} m_3 \left((\dot{q}_3 s_2 + \dot{q}_2 c_2 (d_3+d_2))^2 + \dot{q}_1 c_2 s_1 (d_3+d_2) - \dot{q}_3 c_1 c_2 \right.$$

$$\left. + (\dot{q}_2 c_1 s_2 (d_3+d_2))^2 + \dot{q}_3 s_1 c_2 + \dot{q}_1 c_2 c_1 (d_3+d_2) - \dot{q}_2 s_1 s_2 (d_3+d_2) \right)$$

The total kinetic energy is given by $K = K_1 + K_2 + K_3$.

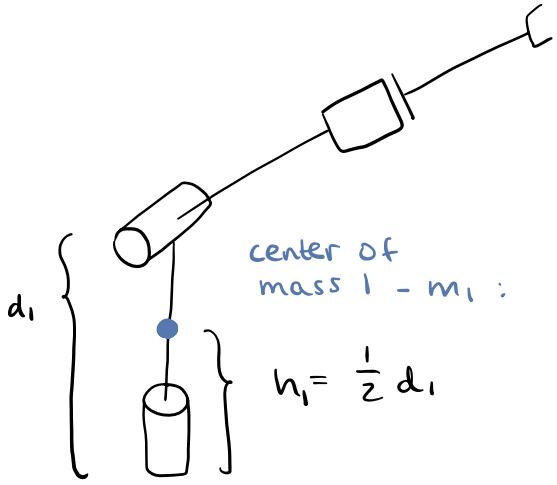
3) Finding the potential energy

→ we use the formula $P = mgh$ to calculate the potential energy for each mass.

→ we use geometry to calculate the height to the point where the mass is centered.

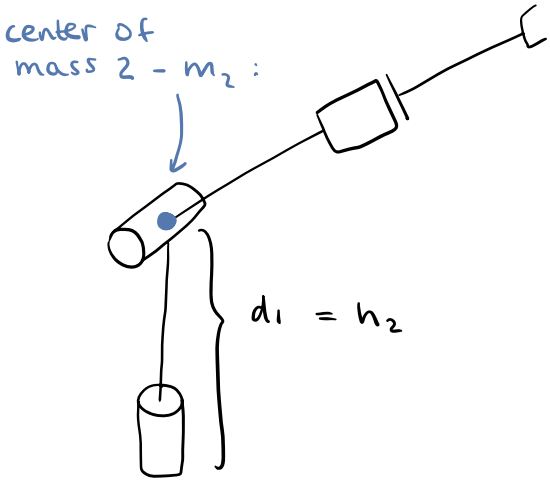
→ g is the gravitational constant

Finding P1: $P_1 = m_1 g h_1$



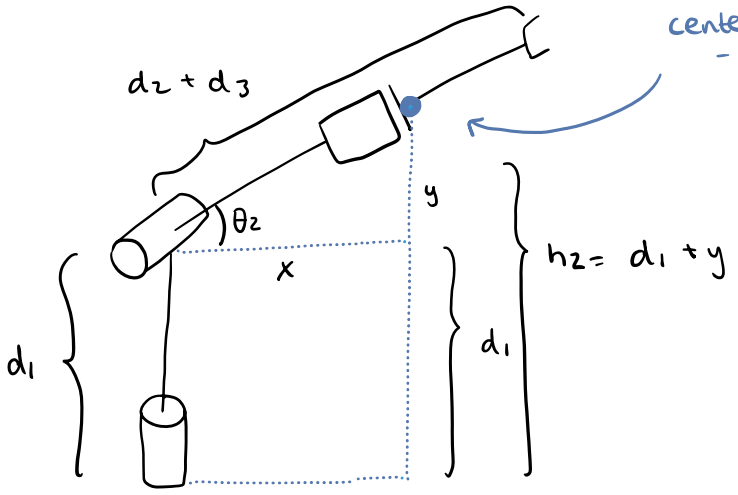
From the drawing we can see that $h_1 = \frac{1}{2} d_1$ so the potential energy is $P_1 = \frac{1}{2} m_1 g d_1$

Finding P2: $P_2 = m_2 g h_2$



From the drawing we can see that $h_2 = d_1$ so the potential energy is $P_2 = m_2 g d_1$

Finding P3: $P_3 = m_3 g h_3$



center of mass 3 - m_3
- middle of the length $d_2 + d_3$

$$\sin\theta_2 = \frac{y}{\frac{1}{2}(d_2+d_3)}$$

$$\Rightarrow y = \frac{1}{2}(d_2+d_3)s_2$$

$$\Rightarrow h_3 = d_1 + \frac{1}{2}(d_2+d_3)s_2$$

$$P_3 = m_3 g h_3 = m_3 g \left(d_1 + \frac{1}{2}(d_2+d_3)s_2\right)$$

Now we can find the total potential energy:

$$P = P_1 + P_2 + P_3 = \frac{1}{2}m_1 g d_1 + m_2 g d_1 + m_3 g \left(d_1 + \frac{1}{2}(d_2+d_3)s_2\right)$$

4) Assembling the Lagrangian term

→ Now that we have found the kinetic and potential energy, we can assemble the Lagrangian:

$$L = K - P = (K_1 + K_2 + K_3) - (P_1 + P_2 + P_3)$$

$$= \frac{1}{2} m_3 \Big((\dot{q}_3 s_2 + \dot{q}_2 c_2 (d_3+d_2))^2 + \dot{q}_1 c_2 s_1 (d_3+d_2) - \dot{q}_3 c_1 c_2$$

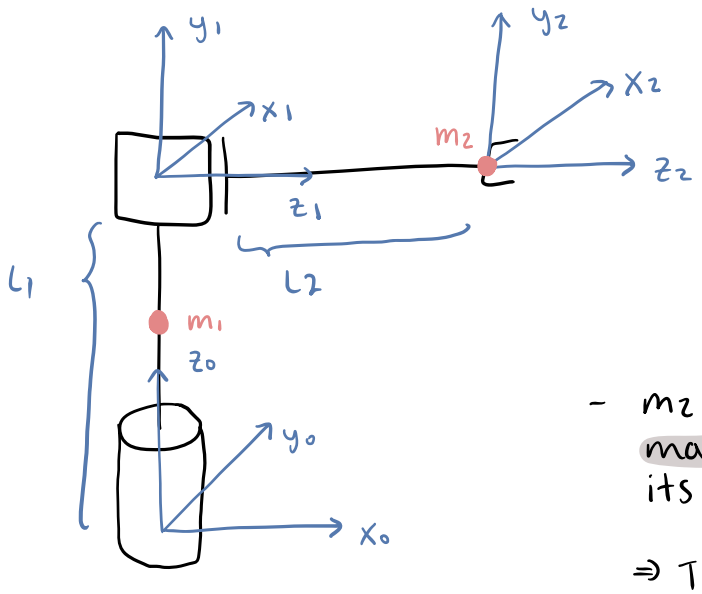
$$+ (\dot{q}_2 c_1 s_2 (d_3+d_2))^2 + \dot{q}_3 s_1 c_2 + \dot{q}_1 c_2 c_1 (d_3+d_2) - \dot{q}_2 s_1 s_2 (d_3+d_2)\Big)$$

$$- \frac{1}{2} m_1 g d_1 - m_2 g d_1 - m_3 g \left(d_1 + \frac{1}{2}(d_2+d_3)s_2\right)$$

That's it! :)

→ Now we have found the Lagrangian and we are one step closer to describing the dynamics of our manipulator. This will be the theme of next weeks group session.

Example - dynamics of a 2DOF robot



- L_1 is a rectangular solid with uniformly distributed mass. This is equivalent to having its mass concentrated in the middle of the link. This mass is m_1.

- m_2 is a point mass. A point mass has no inertia, so its inertia tensor is always 0.

⇒ This means that its rotational kinetic term is always 0, so you don't have to calculate it at all. (So, $K_{w_2} = 0$.).

- The first link is a rectangular solid with uniformly distributed mass m_1. The inertia tensor for m_1 is:

$$I_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{1,zz} \end{bmatrix}$$

(this is usually given)

Task: Find the Lagrangian term.

1) Find T_1^0 and T_2^0.

DH-table:

i	θ_i	d_i	α_i	a_i
1	$\theta_1^* + 90°$	L_1	90°	0
2	0	L_2^*	0	0

$$T_1^0 = A_1 = \begin{bmatrix} -s_1 & 0 & c_1 & 0 \\ c_1 & 0 & s_1 & 0 \\ 0 & 1 & 0 & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_2^* \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^0 = A_1 A_2 = \begin{bmatrix} -s_1 & 0 & c_1 & c_1 L_2^* \\ c_1 & 0 & s_1 & s_1 L_2^* \\ 0 & 1 & 0 & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2) Find the Jacobian

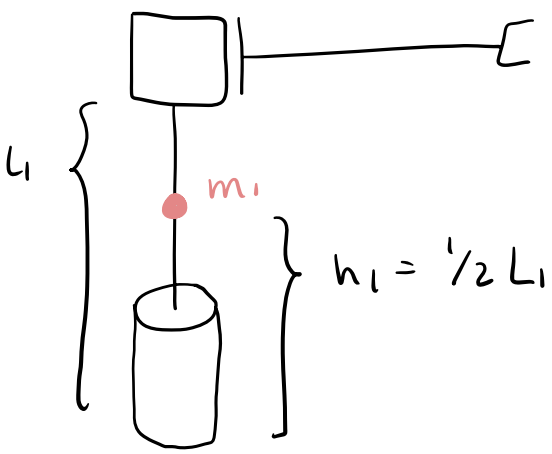
$$J = \begin{bmatrix} J_v \\ J_w \end{bmatrix} \qquad J_v = [\, z_0 \times (o_2 - o_0) \quad z_1 \,] \qquad J_w = [\, z_0 \quad 0 \,]$$

$$J_v = \begin{bmatrix} -s_1 L_2 * & c_1 \\ c_1 L_2 * & s_1 \\ 0 & 0 \end{bmatrix} \qquad J_w = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

3) Calculate the potential and Kinetic energy

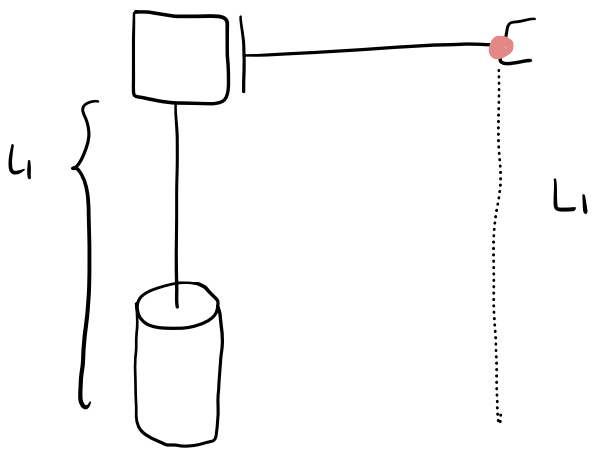
Potential : $P = P_1 + P_2 = m_1 g h_1 + m_2 g h_2$

mass 1:



- m_1 is uniformly distributed, so its at the center of the body/link
- $P_1 = m_1 g h_1 = 1/2\, m_1 g L_1$

mass 2:



- m_2 is a point mass, so the mass center is in that point.
- $P_2 = m_2 g h_2 = m_2 g L_1$

Total potential energy:

$$P = P_1 + P_2 = 1/2 m_1 g L_1 + m_2 g L_1$$

Kinetic :

$$K = K_1 + K_2$$
$$= \tfrac{1}{2} m_1 v_1^T v_1 + \tfrac{1}{2} \omega_1^T I_1 \omega_1 + \tfrac{1}{2} m_2 v_2^T v_2 + \underbrace{\tfrac{1}{2} \omega^T I_2 \omega_2}_{0}$$

↓ will be 0 cause m_2 is a point mass

Components :

$$v_1 = J_{v_1} \dot{q} = \begin{bmatrix} -s_1 \cdot 0 & 0 \\ c_1 \cdot 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{L}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = J_{v_2} \dot{q} = \begin{bmatrix} -s_1 L_2 & c_1 \\ c_1 L_2 & s_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{L}_2 \end{bmatrix} = \begin{bmatrix} -s_1 L_2 \dot{\theta}_1 + c_1 \dot{L}_2 \\ c_1 L_2 \dot{\theta}_1 + s_1 \dot{L}_2 \\ 0 \end{bmatrix}$$

$$\omega_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{L}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

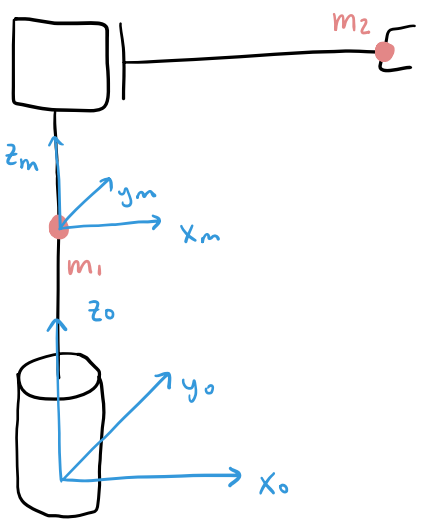
→ There is no point in calculating ω_2 because m_2 is a point mass and has inertia tensor equal to 0, so its angular kinetic velocity component will be 0.

→ The last component is I_1 – the inertia tensor. As described earlier, we have $I_1 = R_1 I_1 R_1^T$.

To compute this, we need to know R_1, while I_1 was given.

→ R_1 is given by assigning a coordinate frame to the mass and finding the rotation from frame 0 to this frame.

⇒ so from the base frame to the mass frame



→ The rotation between frame 0 and frame m is given by $R_{(z,\theta_1)}$, so a rotation around z_0 about an arbitrary angle theta θ_1.

$$\rightarrow R_1 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Calculating $I_1 = R_1 I_1 R_1^T$

$$I_1 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{1,zz} \end{bmatrix} \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{1,zz} \end{bmatrix} \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{1,zz} \end{bmatrix}$$

Now we have all components, so we can calculate K_1 and K_2.

$$K_1 = \tfrac{1}{2} m_1 v_1^T v_1 + \tfrac{1}{2} \omega_1^T I_1 \omega_1$$

$$= \frac{1}{2} m_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{1,zz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 I_{1,zz} \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \tfrac{1}{2} \dot{\theta}_1^2 I_{1,zz}$$

$$K_2 = \tfrac{1}{2} m_2 v_2^T v_2 + \tfrac{1}{2} \omega_2^T I_2 \omega_2$$

$$= \frac{1}{2} m_2 \begin{bmatrix} -s_1 L_2 \dot{\theta}_1 + c_1 \dot{L}_2 \\ c_1 L_2 \dot{\theta}_1 + s_1 \dot{L}_2 \\ 0 \end{bmatrix}^T \begin{bmatrix} -s_1 L_2 \dot{\theta}_1 + c_1 \dot{L}_2 \\ c_1 L_2 \dot{\theta}_1 + s_1 \dot{L}_2 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} m_2 \left(s_1^2 L_2^2 \dot{\theta}_1 - s_1 c_1 L_2 \dot{L}_2 \dot{\theta}_1 - s_1 c_1 L_2 \dot{L}_2 \dot{\theta}_1 + c_1^2 \dot{L}_2^2 \right.$$

$$\left. + c_1^2 L_2^2 \dot{\theta}_1 + s_1 c_1 L_2 \dot{L}_2 \dot{\theta}_1 + s_1 c_1 L_2 \dot{L}_2 \dot{\theta}_1 + s_1^2 \dot{L}_2^2 \right)$$

$$= \frac{1}{2} m_2 \left(L_2^2 \dot{\theta}_1 + \dot{L}_2^2 \right)$$

Total kinetic energy: $K = K_1 + K_2 = \frac{1}{2} \dot{\theta}_1^2 I_{1,zz} + \frac{1}{2} m_2 \left(L_2^2 \dot{\theta}_1 + \dot{L}_2^2 \right)$

The Lagrangian: $L = K - P$

$$\underline{L = \tfrac{1}{2} \dot{\theta}_1^2 I_{1,zz} + \tfrac{1}{2} m_2 \left(L_2^2 \dot{\theta}_1 + \dot{L}_2^2 \right) - \tfrac{1}{2} m_1 g L_1 - m_2 g L_1}$$

Tips for the oblig:

1) start early!!!!

   → There are a lot of calculations and math to do, so you should do yourself a favor and start early!!!

2) The expressions might be long and a "little" bit ugly, so you can use python/matlab in order to check the expressions you got.

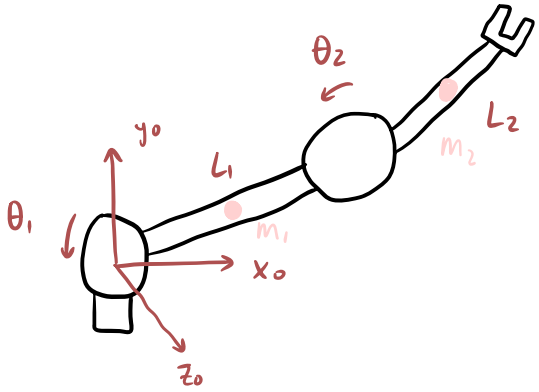
3) Try to do task 1 by hand - this is relevant for the exam as you will most likely have to calculate dynamics for a 2 DOF robot.

4) Don't solve task 2 by hand!! As a previous group teacher said - you will hate yourself beyond recognition :).

   sympy<3 → Use python or matlab here if you want to be happy :)

5) Read the task and the information about the robot (like inertia or mass distribution) CAREFULLY as there are "hints" about what you have to do.

Task - 2-link planar arm



$$J = \begin{bmatrix} -L_1 s_1 - L_2 s_{12} & -L_2 s_{12} \\ L_1 c_1 + c_{12} & c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

- Link mass is uniformly distributed for both
- l_1 and l_2 are 0

Find the lagrangian :)