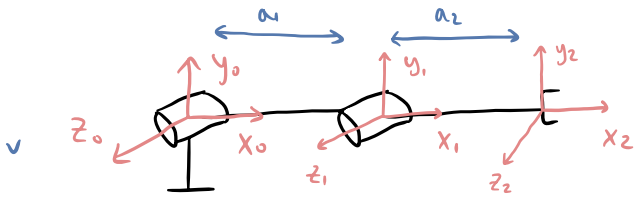
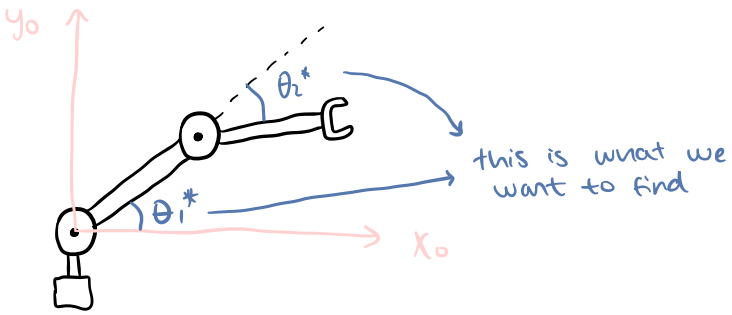
inverse kinematics - geometrical approach



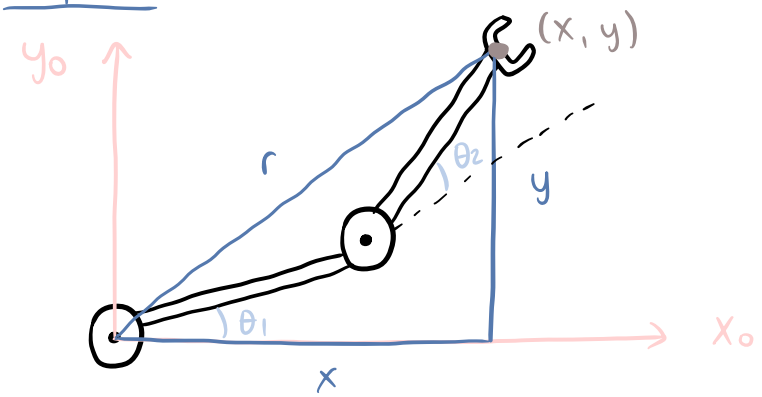
PS. the initial position of the robot (and the frames) matters! And another start position can give a (slightly) different result!

Drawing in $x_0 - y_0$ plane



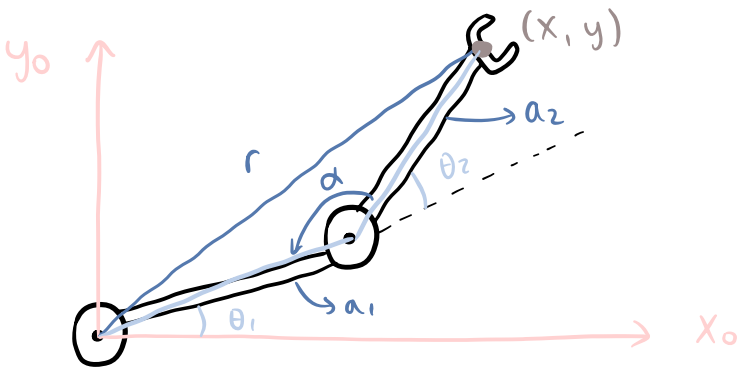
Tips: it can help to define expressions like this using the drawing along the way.

Step 1:



$$r^2 = x^2 + y^2 \quad \text{(pythagoras)}$$

Step 2:



using cosine rule:

$$a^2 = b^2 + c^2 - 2bc\cos(a)$$

$$r^2 = a_1^2 + a_2^2 - 2a_1a_2\cos(\alpha)$$

$$\cos(\alpha) = \frac{a_1^2 + a_2^2 - r^2}{2a_1a_2}$$

$$= \frac{a_1^2 + a_2^2 - x^2 - y^2}{2a_1a_2}$$

$\theta_2 = \pi - \alpha \Rightarrow \alpha = \pi - \theta_2 \Rightarrow \cos(\alpha) \Rightarrow$
$\cos(\pi - \theta_2) = -\cos(\theta_2) \Rightarrow$
$\cos(\alpha) = -\cos(\theta_2)$

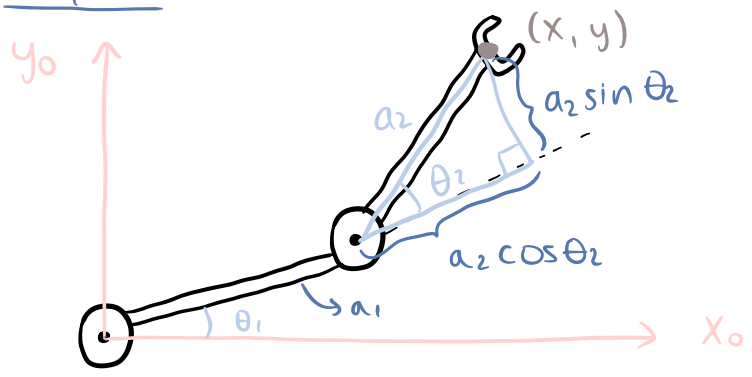
We then have that

$$\cos(\theta_2) = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1a_2}$$:D

call this D

The point is: we have an expression for $\cos\theta_2$, so we want then to find an expression for $\sin\theta_2$ so that we can find $\tan\theta_2$ and then solve using atan2 !

Step 3:



$$a_2^2 = a_2^2\cos^2\theta_2 + a_2^2\sin^2\theta_2$$

$$\Rightarrow \cos^2\theta_2 + \sin^2\theta_2 = 1$$

$$\Rightarrow \sin\theta_2 = \pm\sqrt{1 - \cos^2\theta_2}$$

$$\Rightarrow \sin\theta_2 = \pm\sqrt{1 - D^2}$$

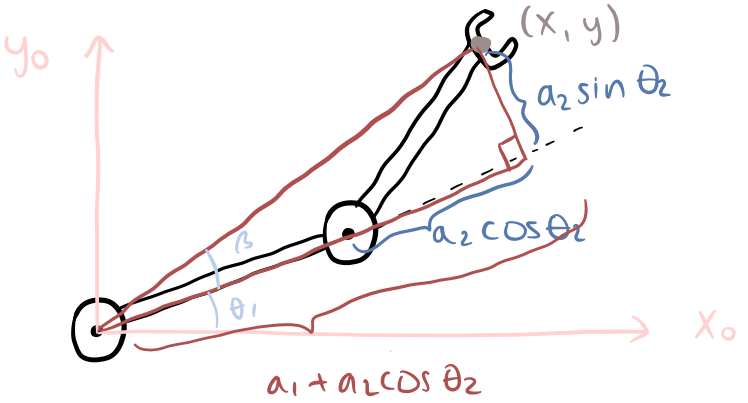
↳ use this one a lot in the analytical approach too.

Now we find $\tan\theta_2$ so that we can use atan2:

$$\tan\theta_2 = \frac{\sin\theta_2}{\cos\theta_2} = \frac{\pm\sqrt{1 - D^2}}{D}$$

$$\Rightarrow \theta_2 = \text{atan2}\left(\frac{\pm\sqrt{1 - D^2}}{D}\right)$$

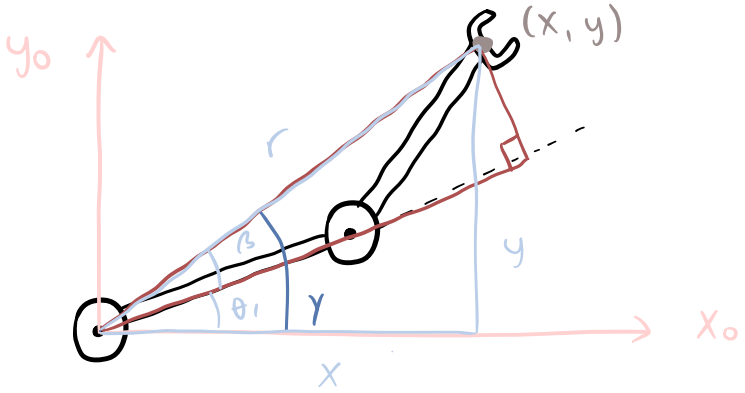
Step 4:



$$\tan\beta = \frac{a_2\sin\theta_2}{a_1 + a_2\cos\theta_2} =$$

$$\beta = \text{atan2}(a_2\sin\theta_2, a_1 + a_2\cos\theta_2)$$

Step 5:



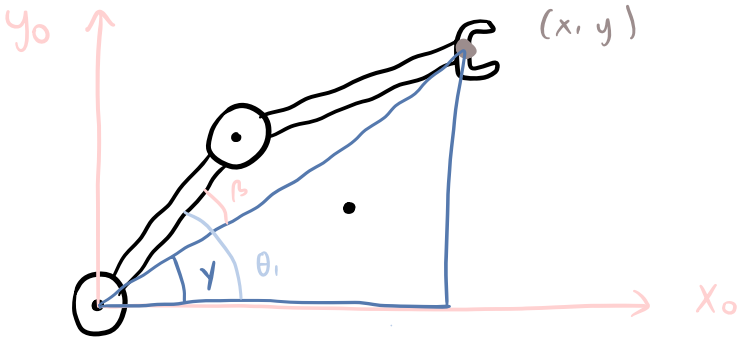
$$\tan\gamma = \frac{r\sin\gamma}{r\cos\gamma} = \frac{y}{x}$$

$$\gamma = \text{atan2}(y, x)$$

$$\theta_1 + \beta = \gamma \Rightarrow \theta_1 = \gamma - \beta$$

$$\theta_1 = \text{atan2}(y, x) - \text{atan2}(a_2\sin\theta_2, a_1 + a_2\cos\theta_2)$$

This is one solution for the inverse kinematics – the elbow down solution, but we have another solution – the elbow up solution.



$$\theta_1 = \gamma + \beta$$

$$\theta_1 = \text{atan2}(y, x) + \text{atan2}(a_2\sin\theta_2, a_1 + a_2\cos\theta_2)$$

Solutions for inverse kinematics for this robot:

$$\theta_1 = \text{atan2}(y, x) \pm \text{atan2}(a_2\sin\theta_2, a_1 + a_2\cos\theta_2)$$

$$\theta_2 = \text{atan2}\left(\frac{\pm\sqrt{1 - D^2}}{D}\right), \quad D = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1a_2}$$