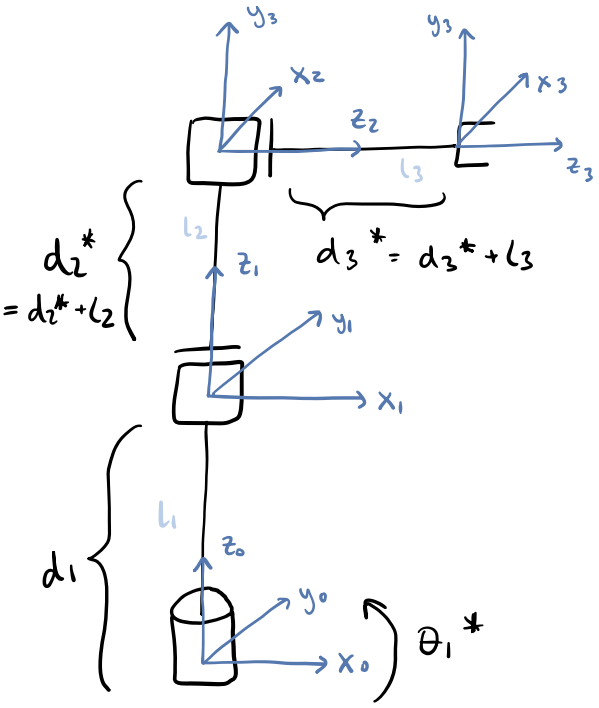
Inverse Kinematics part 1

Repetition forward kinematics

Forward kinematics: You have the angles or length (variables) of the joints and you want to know where the tooltip ends in space.

Example - calculating forward kinematics of a robot:



→ cylindrical manipulator

1) Add coordinate frames to the drawing

Rules:
- z in the direction of action
- x_i 90° to z_{i-1}
- x_i needs to intersect z_{i-1}
  → if not, we move the origin o_i, so they intersect.
- y from the right hand rule

2) Fill in the DH-table

		θ_i	d_i	a_i	α_i
links	1	θ_1^*	d_1	0	0
	2	90°	d_2^*	0	90°
	3	0	d_3^*	0	0

→ θ_i: angle of rotation for a revolute joint
- angle from x_{i-1} to x_i about z_{i-1}

→ d_i: distance moved for a prismatic joint
- distance from o_{i-1} to intersection of x_i and z_{i-1}

→ a_i: distance from intersection of x_i and z_{i-1} to o_i

→ α_i: angle from z_{i-1} to z_i measured about x_i

→ essentially we want to find the transformations between these 4 frames.

→ goal: find transformation between base frame and end-effector frame:

$$H_3^0 = H_1^0 H_2^1 H_3^2$$

3) For each link 1, 2, 3 we use the "special matrix" and fill in all the values θ_i, d_i, a_i, α_i that we found:

$$A_i = \begin{bmatrix} C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i & a_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{the special matrix}$$

$$A_1 = H_1^0 = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & 0 \\ S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = H_2^1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = H_3^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4) Find transformation from base frame to tool tip frame

- In forward kinematics we essentially want to find the transformation from base frame to tooltip frame, because we want to know where the tooltip is with respect to the base.
- We want to find H_3^0 - remember $H_y^x = H_z^x H_y^z$.

$$H_3^0 = H_1^0 H_2^1 H_3^2$$

$$= \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & 0 \\ S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -S\theta_1 & 0 & C\theta_1 & 0 \\ C\theta_1 & 0 & S\theta_1 & 0 \\ 0 & 1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -S\theta_1 & 0 & C\theta_1 & C\theta_1 d_3 \\ C\theta_1 & 0 & S\theta_1 & S\theta_1 d_3 \\ 0 & 1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{the forward kinematics}$$

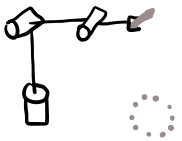
Inverse Kinematics

Inverse Kinematics: Given the position of the tooltip, what are the joint variables θ_i and d_i?

→ more useful than forward kinematics, because we often know the point we want to reach (where we want the tool tip to be), but not how to get there (what joint angles and extentions we have to use in order to get there.

→ more difficult (it requires thinking outside the box)

→ Example (from oblig 5): you want to use a manipulator to draw a circle on a paper. You can calculate some points that together define a circle. To draw the circle, the tip of the manipulator has to pass through all these points.



We can use inverse kinematics to calculate the angles of the joints of the manipulator needed to reach each point. These angles will then be sent as commands to the manipulator in order to control it.

→ this is what you'll do in oblig 5.

Method 1 – the analytic approach → recommended :)

→ The idea is to set up a system of equations and solve for the joint variables (θ_i and d_i) → we end up with expressions for θ_i and d_i.

→ We want to use the expression for the position of the tooltip, which corresponds to the **last column of our forward kinematics matrix** (it is a translation vector (x, y, z)).

→ This vector is the position of the tool tip in the base frame and that's what we use to calculate the inverse kinematics.

→ we set up a set of equations on the form:

$x = a$
$y = b$
$z = c$

translation vector from forward kinematics matrix:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \Big| \Big| \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

is the point we want to reach and is known

→ the goal is to **put the variables on the left hand side of the equation**: $\theta_i = \ldots$ and $d_i = \ldots$

→ we do this by manipulating the expressions algebraically

Some useful formulas to use:

Double Angle Identities

$\sin(2\theta) = 2\sin\theta\cos\theta$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$\cos(2\theta) = 2\cos^2\theta - 1$

$\cos(2\theta) = 1 - 2\sin^2\theta$

Half Angle Indentities

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Sum to Product of Two Angles

$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

Product to Sum of Two Angles

$$\sin\theta\sin\phi = \frac{[\cos(\theta-\phi) - \cos(\theta+\phi)]}{2}$$

$$\cos\theta\cos\phi = \frac{[\cos(\theta-\phi) + \cos(\theta+\phi)]}{2}$$

$$\sin\theta\cos\phi = \frac{[\sin(\theta+\phi) + \sin(\theta-\phi)]}{2}$$

$$\cos\theta\sin\phi = \frac{[\sin(\theta+\phi) - \sin(\theta-\phi)]}{2}$$

Sum/Difference Indentities

$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$

$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$

Example - calculating the inverse kinematics with analytical appr.

From the previous example, we have the forward kinematics:

$$H_3^0 = \begin{bmatrix} -S\theta_1 & 0 & C\theta_1 & C\theta_1 d_3 \\ C\theta_1 & 0 & S\theta_1 & S\theta_1 d_3 \\ 0 & 1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ position of tooltip in base frame

From this, we get the equations

$$\left.\begin{aligned} x &= C\theta_1^* d_3^* \\ y &= S\theta_1^* d_3^* \\ z &= d_1 + d_2^* \end{aligned}\right\}$$

To solve these, we use the trigonometric formulas and some creativity

→ our variables are θ_1, d_2 and d_3, and d_1, l_1, l_2, l_3 are known.

1) Finding d_2^*: * starting with the most obvious one *

$$z = d_1 + d_2^* \Rightarrow \underline{d_2^* = z - d_1}$$

→ both z and d_1 are known

2) Finding θ_1^*:

$$\left.\begin{aligned} x &= C\theta_1^* d_3^* \\ y &= S\theta_1^* d_3^* \end{aligned}\right\} \Rightarrow \frac{y}{x} = \frac{S\theta_1^* \cancel{d_3^*}}{C\theta_1^* \cancel{d_3^*}} = \frac{S\theta_1^*}{C\theta_1^*} = \tan\theta_1^*$$

$$\tan\theta_1^* = \frac{y}{x} \Rightarrow \theta_1^* = \arctan\left(\frac{y}{x}\right)$$

$$\rightarrow \underline{\theta_1^* = \operatorname{atan2}(y, x)}$$

Tips 1: look for common terms in the expressions that you can cross out

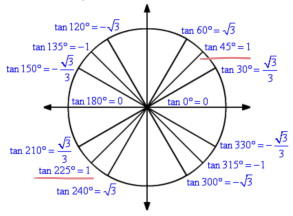
→ Note: we use the atan2 function (instead of the normal arctangent function).

$$\operatorname{atan2}(y,x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0, \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \geq 0, \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0, \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

(different programming languages may have different implementations
→ in python its atan2(y,x) which is the arctangent of y/x)

atan2 is the 4 quadrant atan - allows us to know in which of the quadrants the result is in and is defined for all values.

→ normal because the tangent is the same for 2 quadrants:



tan 45° = tan 225° = 1
so if tan = 1, we don't know what angle we have → we need atan2.

3) Finding d_3^*

$$x^2 + y^2 = (c_1 d_3^*)^2 + (s_1 d_3^*) = c_1^2 d_3^{*2} + s_1^2 d_3^{*2}$$

$$= d_3^{*2}(c_1^2 + s_1^2) = d_3^{*2}$$

$$\Rightarrow d_3^* = \sqrt{x^2 + y^2}$$

Tips 2: calculate x^2+y^2 and reduce by using trig formulas like $\cos^2 + \sin^2 = 1$

So, we have the inverse kinematics:

$$\theta_1^* = \operatorname{atan2}(y,x)$$
$$d_2^* = z - d_1$$
$$d_3^* = \sqrt{x^2+y^2}$$

so if you have a point within the boundary you can use these formulas to find the joint angles that will get your manipulator there

PS: the IK problem may not have a solution, and the solutions might violate joint limits and might not be unique

PS: you don't have to find the expressions in any particular order but if one expression (1) depends on another expression (2), you have to find (1) first and then (2).

↓
but here we are just concerned with finding expressions for the joint variables but in practice we can apply different methods to get an actually usable solution

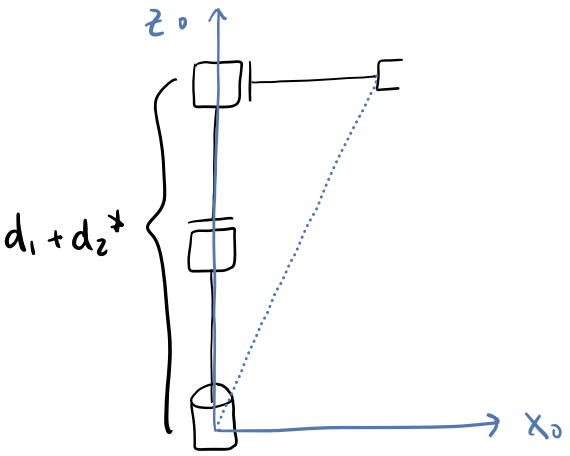
Method 2 - geometric approach

→ the idea is to project the manipulator in various planes and pretend it's in 2D.

→ we look at the manipulator in side view
→ we look at the manipulator from top view
→ and then identify as many equations as possible using trigonometry.

Example - finding the inverse Kinematic equations using the geometric approach.

1) Projecting onto the side view plane (x_0, z_0-plane)

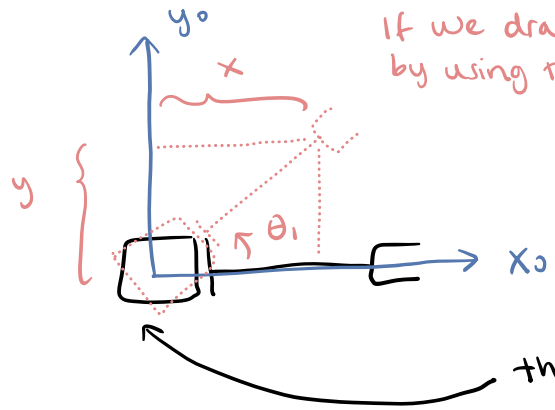


→ The height of the tooltip (which is in the z-direction) is given by the length of link 1 (d_1) and the variable length of link 2 (d_2^*) so:

$$z = d_1 + d_2^* \quad \Rightarrow \quad d_2^* = z - d_1$$

- Note: We cannot find θ_1 from this drawing, cause rotating θ_1 would cause the tooltip to move in and out of the plane, but that's not possible since we're in 2D.

2) Projecting onto the top view plane (x_0, y_0-plane)



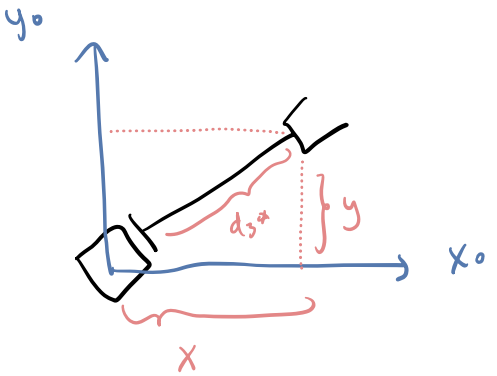
If we draw it like this we can see that we can find θ_1 by using the tangent.

→ $$\tan\theta_1 = \frac{\sin\theta_1}{\cos\theta_1} = \frac{y}{x}$$

$$\Rightarrow \quad \theta_1 = \arctan\left(y/x\right)$$

$$\theta_1 = \text{atan2}(y, x)$$

the joints are stacked over eachother



→ Using pythagoras: $c^2 = a^2 + c^2$

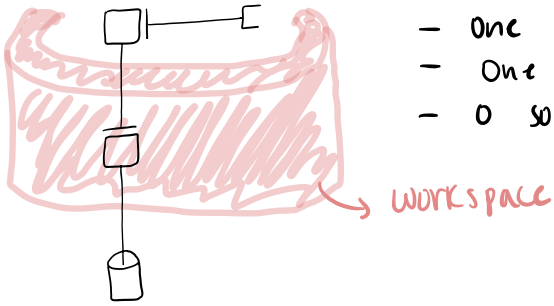
$$x^2 + y^2 = d_3^{*2} \quad \Rightarrow \quad d_3^* = \sqrt{x^2 + y^2}$$

Ps: the solution of the geometrical approach relies on the start configuration!

Related to inverse kinematics you might get asked about the number of solutions that exist for a manipulator.

→ to do this we can analyse the manipulator and visualise how it would reach points inside the workspace, outside the workspace and on the boundary of the workspace.

→ In the example above we have:



- one solution within the workspace
- one solution inside the workspace
- 0 solutions outside the workspace

Singularities: configurations where the orientation and position of the end effector cannot be uniquely determined by the joint parameters. This leads to loss of DOFs so that the robot cannot move in certain directions.

→ Near a singular configuration, the inverse kinematics can either give infinitely many solutions or no solution at all because at a singularity the relationship between the joint movements and end effector movements becomes undetermined or very sensitive.

→ we will come back to this in the velocity kinematics part :)