REPETITION

- forward kinematics
- inverse kinematics
- velocity kinematics

FORWARD KINEMATICS

Forward kinematics: find the end effector given the joint variables.

We use the **DH convention** → it is a recipe for finding the forward kinematics.

Recipe

1) Draw coordinate frames on the manipulator

The rules are:

- z in the direction of action



- x according to the rules
  - $x_i \perp z_{i-1}$
  - x_i needs to intersect z_{i-1}
    - move o_i if it doesn't
- y according to right hand rule

Tips:

→ when setting the frames, try to have as little change between the frames as possible, because it leads to a simpler DH table.

2) Fill in the DH table

links	θ_i	d_i	a_i	α_i
1				
⋮				
n				

θ_i : angle from x_{i-1} to x_i measured about z_{i-1}

d_i : distance from O_{i-1} to intersection of x_i and z_{i-1}

a_i : distance from intersection of x_i and z_{i-1} to O_i

α_i : angle from z_{i-1} to z_i measured about x_i

3) Use the DH table to fill in the special matrix for each link / joint

$$A_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ this gives the transformation between 2 frames that come after each other

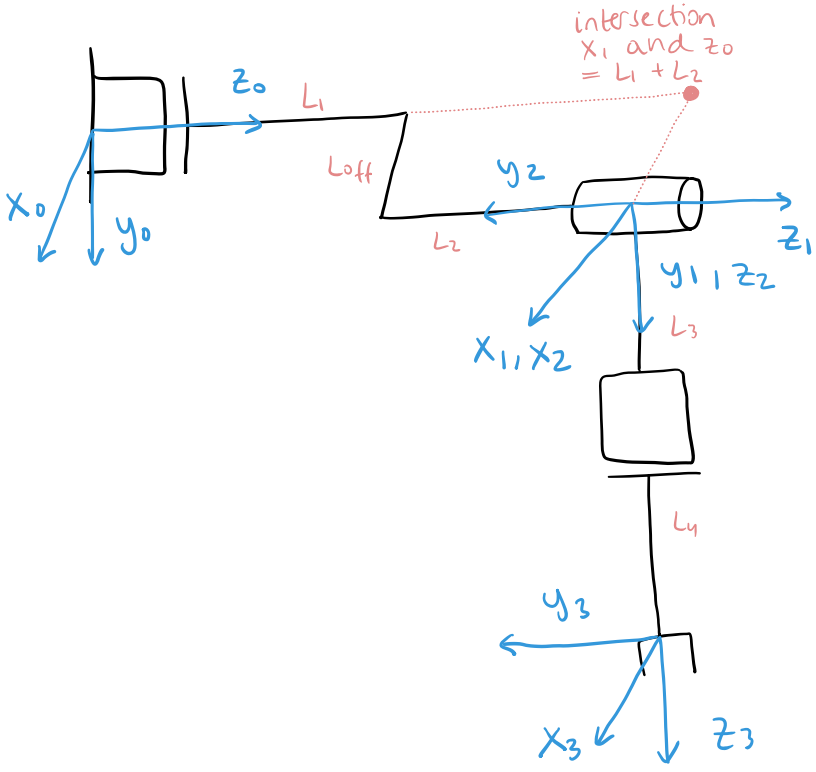
4) Find the transformation between the base and each of the frames by using this formula:

$$H^x_y = H^x_z \, H^z_y$$

→ the forward kinematics is the transformation between the base frame and the tooltip frame

→ the position of the tooltip is given by the last column (3 first rows) of the forward kinematics matrix

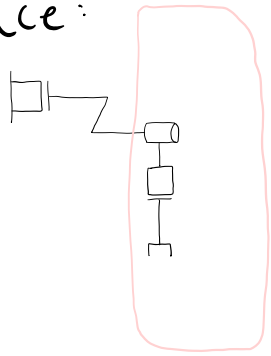
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a) Sketch workspace and classify robot according to its joints:

→ It's a PRP robot.

→ workspace:



b) Assign coordinate frames to the robot according to DH convention and write the DH table.

DH - table:

links	θ_i	d_i	a_i	α_i
1	0	$\tilde{d}_1$	L_{off}	0
2	θ_2^*	0	0	$-90°$
3	0	$\tilde{d}_3$	0	0

$$\tilde{d}_1 = \underbrace{L_1 + L_2}_{\text{fixed length of links}} + \underbrace{d_1^*}_{\text{variable lenght of joint 1/3}}$$

$$\tilde{d}_3 = \underbrace{L_3 + L_4}_{\text{fixed length of links}} + \underbrace{d_3^*}_{\text{variable lenght of joint 1/3}}$$

c) Derive the forward kinematics

→ finding $A_1 = H_1^0$, $A_2 = H_2^1$, $A_3 = H_3^2$ using the special matrix

$$A_1 = H_1^0 = \begin{bmatrix} 1 & 0 & 0 & L_{off} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \tilde{d}_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = H_2^1 = \begin{bmatrix} C_2 & 0 & -S_2 & 0 \\ S_2 & 0 & C_2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = H_3^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \tilde{d}_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ Finding H_1^0, H_2^0 and H_3^0 (forward kinematics)

$$H_1^0 = A_1 = \begin{bmatrix} 1 & 0 & 0 & L_{off} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \tilde{d}_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^0 = H_1^0 H_2^1 = A_1 A_2$$

$$= \begin{bmatrix} 1 & 0 & 0 & L_{off} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \tilde{d}_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_2 & 0 & -S_2 & 0 \\ S_2 & 0 & C_2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_2 & 0 & -S_2 & L_{off} \\ S_2 & 0 & C_2 & 0 \\ 0 & -1 & 0 & \tilde{d}_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3^0 = H_2^0 H_3^2 = A_1 A_2 A_3$$

$$= \begin{bmatrix} C_2 & 0 & -S_2 & L_{off} \\ S_2 & 0 & C_2 & 0 \\ 0 & -1 & 0 & \tilde{d}_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \tilde{d}_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_2 & 0 & -S_2 & -\tilde{d}_3 S_2 + L_{off} \\ S_2 & 0 & C_2 & C_2 \tilde{d}_3 \\ 0 & -1 & 0 & \tilde{d}_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

INVERSE KINEMATICS

Inverse kinematics: Find the joint variables (angles or distances) based on the tool tip position

→ We can use the geometric approach where we look at the robot from different planes and use geometry to find expressions for the joint variables

→ Or we can use the analytic approach where we set up n (= nr of joint variables) equations and try to solve for the joint variables

Recipe analytic approach:

1) Take the translation vector part from the forward kinematics (which describes the position of the tool tip) and set up equations like this:

x = x-part of translation vector

y = y-part of translation vector

z = z-part of translation vector

2) Then use these to solve for the joint variables by using trigonometric rules and algebra rules.

Recipe geometric approach:

1) Draw the robot and relevant frames from different perspectives and pretend it's 2D.

- project in side view (where we look at the manipulator from its side)
- project in top view (where we look at the robot from above).

2) Use geometric rules to find expressions for the joint variables

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e) Derive the inverse kinematics of the robot

Analytic approach:

$$\begin{aligned} x &= -\tilde{d}_3 s_2 + l_{off} &&= -(L_2 + L_3 + d_3^*) s_2 + l_{off} \\ y &= c_2 \tilde{d}_3 &&= (L_2 + L_3 + d_3^*) c_2 \\ z &= \tilde{d}_1 &&= L_1 + L_2 + d_1^* \end{aligned}$$

1) solving for d_1^*:

$$z = L_1 + L_2 + d_1^* \Rightarrow d_1^* = z - L_1 - L_2$$

2) solving for d_3^*:

$$\begin{aligned} x &= -\tilde{d}_3 s_2 + l_{off} \\ y & \quad c_2 \tilde{d}_3 \end{aligned} \Rightarrow \tilde{d}_3 s_2 = l_{off} - x$$

→ we see that if we square both sides we get c_2^2 and s_2^2 and we can then use $c_2^2 + s_2^2 = 1$

$$(l_{off} - x)^2 = (s_2 \tilde{d}_3)^2 = s_2^2 \tilde{d}_3$$

$$y^2 = (c_2 \tilde{d}_3)^2 = c_2^2 \tilde{d}_3^2$$

$$(l_{off} - x)^2 + y^2 = s_2^2 \tilde{d}_3 + c_2^2 \tilde{d}_3^2$$

$$= \tilde{d}_3^2 (c_2^2 + s_2^2) = \tilde{d}_3^2 \Rightarrow$$

$$\tilde{d}_3 = \sqrt{(l_{off} - x)^2 + y^2} \Rightarrow$$

$$L_3 + L_4 + d_3^* = \sqrt{(l_{off} - x)^2 + y^2} \Rightarrow$$

$$d_3^* = \sqrt{(l_{off} - x)^2 + y^2} - L_3 - L_4$$

(Alternatively, $d_3^* = y/c_2 - L_3 - L_4$, since θ_2 doesn't depend on d_3^*).

3) solving for θ_2 * :

$$\begin{matrix} x \\ y \end{matrix} = \begin{matrix} -\tilde{d}_3 s_2 + loff \\ c_2 \tilde{d}_3 \end{matrix} \Rightarrow \quad loff - x = s_2 \tilde{d}_3$$

→ we know that sin / cos = tan, so we can calculate this and get rid of $\tilde{d}_3$ and be left with θ_2 only:

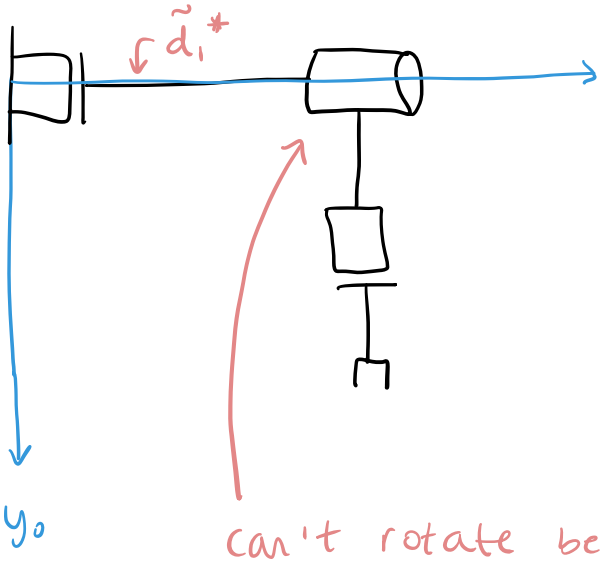
$$\frac{s_2 \tilde{d}_3}{c_2 \tilde{d}_3} = \frac{loff - x}{y} \Rightarrow t_2 = \frac{loff - x}{y}$$

→ We use atan 2 as usual:

$$\theta_2 = atan2(y, loff - x)$$

Geometric approach:

→ side view → (y_0, z_0) - plane

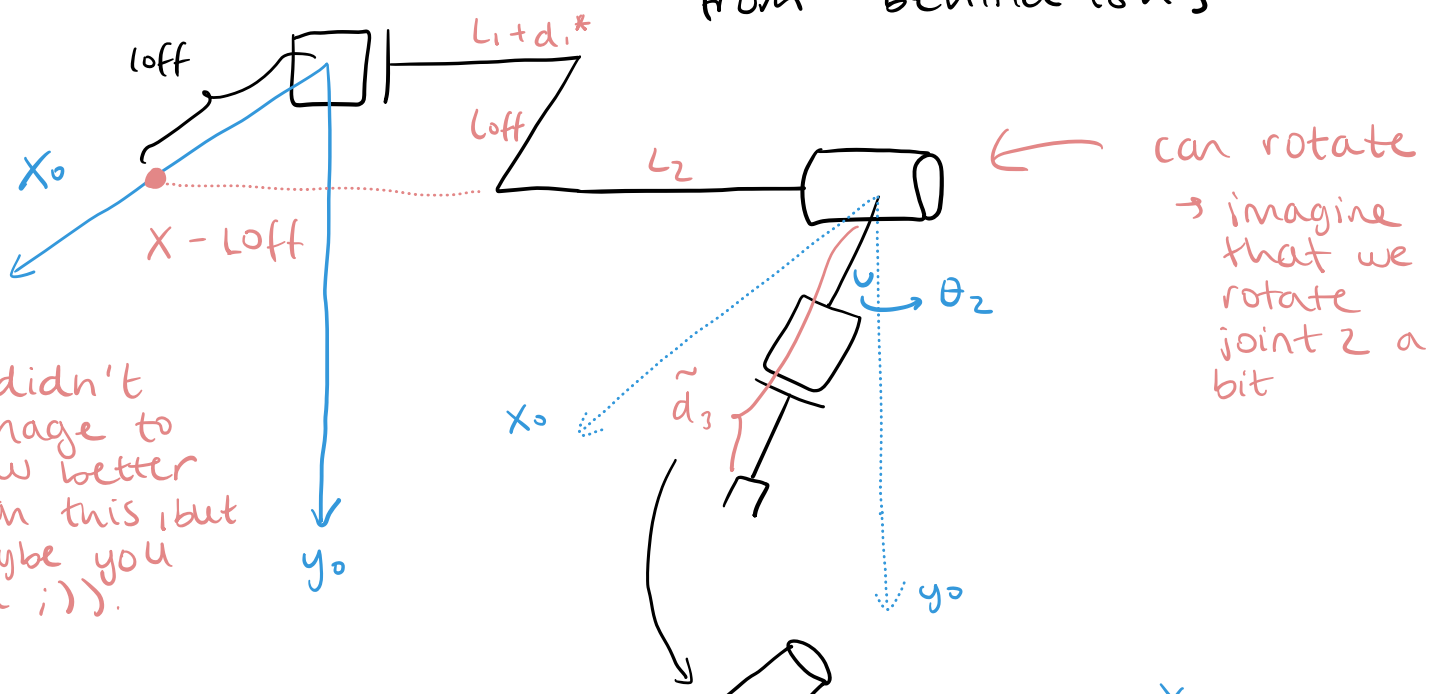


→ the z-coordinate of the tool tip depends on how extended the first joint is - so it's length

$\Rightarrow \underline{\tilde{d}_1 = z}$

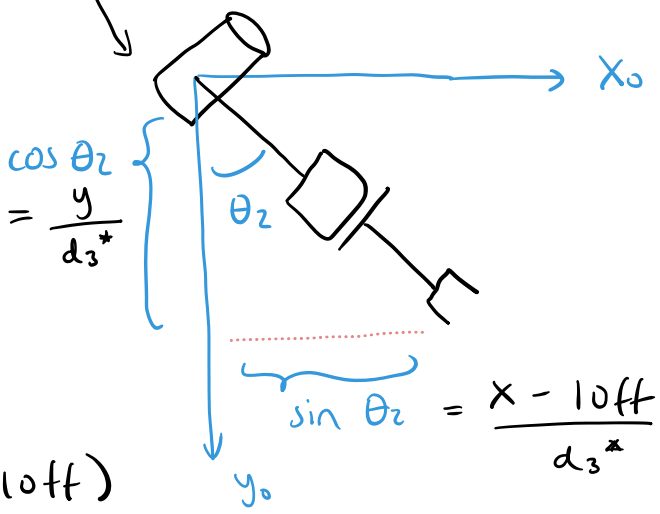
can't rotate because it's only 2D and a rotation would go outside of the plane

→ (x_0, y_0)-plane (looking at the robot from behind ish)



can rotate
→ imagine that we rotate joint 2 a bit

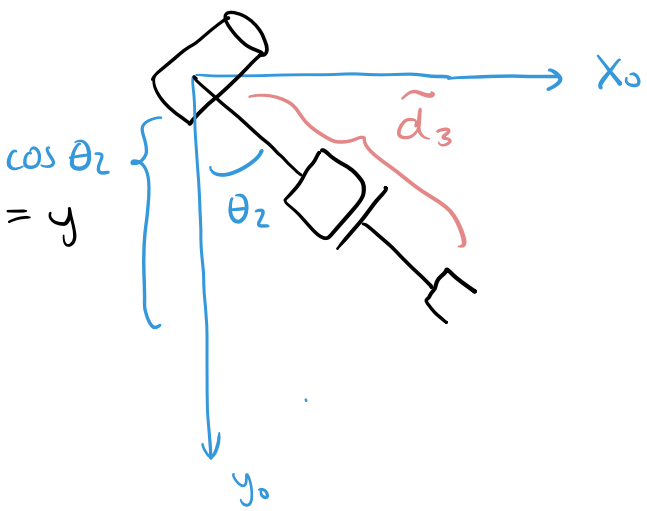
(i didn't manage to draw better than this, but maybe you can ;)).



$$\Rightarrow \tan\theta_2 = \frac{\sin\theta_2}{\cos\theta_2}$$

$$\Rightarrow \tan\theta_2 = \frac{x - loff}{y}$$

$$\Rightarrow \theta_2 = \operatorname{atan2}(y, x - loff)$$



$$\Rightarrow \cos\theta_2 = \frac{y}{\tilde{d}_3}$$

$$\Rightarrow \tilde{d}_3 = \frac{y}{\cos\theta_2}$$

VELOCITY KINEMATICS

Velocity Kinematics : finding the velocity of the tool tip with respect to the base frame.

→ to find the velocity of the tool tip we need the velocities of all previous joints

→ to deal with velocities, we need the jacobian matrix

→ velocity of tool tip is given by:

$$\xi = J\dot{q} \qquad \rightarrow J \text{ is jacobian} \qquad \left(J = \begin{bmatrix} J_v \\ J_w \end{bmatrix}\right)$$

→ joint velocity:

$$v = J_v \dot{q} \rightarrow J_v \text{ is the linear part of the jacobian}$$

$$w = J_w \dot{q} \rightarrow J_w \text{ is the angular part of the jacobian}$$

Recipe for finding the jacobian:

1) Set up a scheleton of the jacobian first using the rules below

2) Find J_v and J_w

Since the jacobian can be written as $J = \begin{bmatrix} J_v \\ J_w \end{bmatrix}$, we can find J_v and J_w separately.

- J_v → the linear velocity part

we use these to find one and one column in J_v and J_w

$$J_v = \begin{cases} z_{i-1} \times (o_n - o_{i-1}) & \text{for revolute joints} \\ z_{i-1} & \text{for prismatic joints} \end{cases}$$

- J_w → the angular velocity part

$$J_w = \begin{cases} z_{i-1} & \text{for revolute joints} \\ 0 & \text{for prismatic joints} \end{cases}$$

3) Assemble the jacobian according to $J = \begin{bmatrix} J_v \\ J_w \end{bmatrix}$

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d) Derive the Jacobian for the robot

Scheleton:

$$J = \begin{bmatrix} z_0 & z_1 \times (O_3 - O_1) & z_2 \\ 0 & z_1 & 0 \end{bmatrix} \begin{matrix} \rightarrow Jv \\ \rightarrow Jw \end{matrix}$$

z_0: prismatic, z_1: revolute, z_2: prismatic

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad z_2 = \begin{bmatrix} -s_2 \\ c_2 \\ 0 \end{bmatrix}$$

→ find these from $H_1^0(z_1)$, $H_2^0(z_2)$, $H_3^0(z_3)$

$$O_3 - O_1 = \begin{bmatrix} -\tilde{d}_3 s_2 + 10ft \\ \tilde{d}_3 c_2 \\ \tilde{d}_1 \end{bmatrix} \begin{bmatrix} 10ft \\ 0 \\ \tilde{d}_1 \end{bmatrix} = \begin{bmatrix} -\tilde{d}_3 s_2 \\ \tilde{d}_3 c_2 \\ 0 \end{bmatrix}$$

$$z_1 \times (O_3 - O_1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -\tilde{d}_3 s_2 \\ \tilde{d}_3 c_2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -c_2 \hat{d}_3 \\ -s_2 \hat{d}_3 \\ 0 \end{bmatrix}$$

Jacobian:

$$J = \begin{bmatrix} 0 & -c_2 \hat{d}_3 & -s_2 \\ 0 & -s_2 \hat{d}_3 & c_2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Singularities: Positions of the robot where minor movement could require a loot of force to move, thus movement in specific directions can be impossible.

→ Usually singularities exist when the tooltip is on the boundary of the workspace.

→ We can use the determinant of the Jacobian to calculate the singular configurations of a robot:

$$|J_v| = 0$$

HOMOGENEOUS TRANSFORMS

The idea of homogeneous transforms is to be able to represent the movement of the robot with respect to different coordinate frames.

- Usually we have a base frame, a world frame and a task frame.

"Recipe" for using homogeneous transforms

1) If you have multiple frames, identify the frame with coordinates (0,0,0) and use this as an "intermediate stopping station" between the frame where you "are" and the frame you want to "go" to.

   - This is because you know the translation vectors from the (0,0,0) - coordinate frame to your other frames.

   - You can then use the formula:

$$H_y^x = H_z^x H_y^z$$

   - Inverting a homogeneous transform:

$$H_x^y = (H_y^x)^{-1} = \begin{bmatrix} (R_y^x)^T & -(R_y^x)^T d_y^x \\ 0 & 1 \end{bmatrix}$$

2) Find the rotation between two and two frames and the translation vector between them to assemble the homogeneous transform:

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}$$

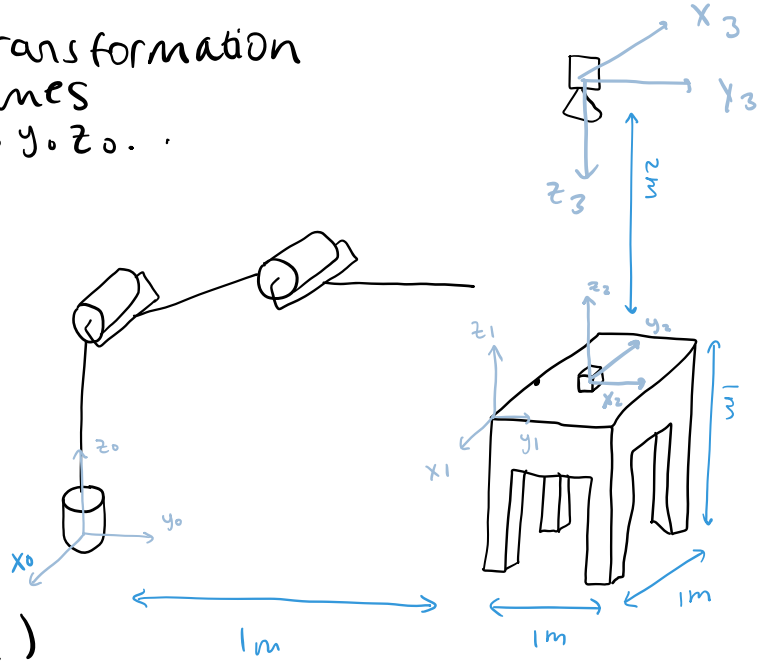
(and eventually multiply according to above rule)

- use right hand rule(s) for the rotation
- post-multiply rotation matrices if the rotation between frames happens with respect to more than 1 frame.

Exercise from the book:

1) Find the homogeneous transformation relating each of these frames to the base frame $0_0 x_0 y_0 z_0$.

$$H_1^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



For finding H_2^0 and H_3^0 we have 2 choices:

- Find H_2^1 and use it to find H_2^0 ($H_2^0 = H_1^0 H_2^1$) and corresponding technique for H_3^0.
- Find H_2^0 and H_3^0 directly. In this case its easy, but that's not always the case.

→ cube side = 20 cm in the center of the table and frame is in the middle

→ table is a 1x1 m square

$$H_2^0 = \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3^0 = \begin{bmatrix} -1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

↳ rotation by π around y-axis

2) Find the homogeneous transformation relating the frame $O_2 X_2 y_2 z_2$ to $O_3 X_3 y_3 z_3$.

$$H_3^2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1.9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From the figure, we can see that from frame 2 to 3, we move up along the z-axis by 1.9 m and rotate our frame by 180° around the current y-axis.