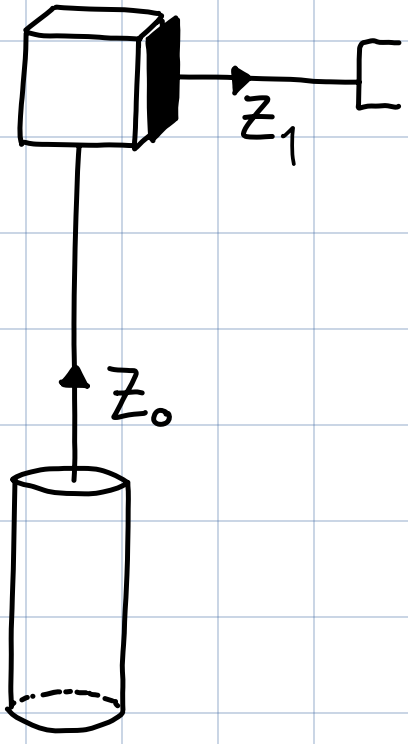


$$J = \begin{bmatrix} -S_1 L_2^* & C_1 \\ C_1 L_2^* & S_1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Try to find the expression yourself (Or look at group session slides)

$$\mathcal{L} = \frac{1}{2} I_{1,zz} \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(L_2^{*2} \dot{\theta}_1^2 + \dot{L}_2^{*2} \right) - \left(\frac{m_1}{2} + m_2 \right) g L_1$$

Write the dynamic equation on the form

$$\tau = D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q)$$

$$\tau_i = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i}$$

(by calculating the partial derivatives)

$$L = \frac{1}{2} I_{1,zz} \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(L_2^{*2} \dot{\theta}_1^2 + \dot{L}_2^{*2} \right) - \left(\frac{m_1}{2} + m_2 \right) g L_1$$

$$L = \frac{1}{2} I_{1,zz} \dot{\theta}_1^2 + \frac{1}{2} m_2 L_2^{*2} \dot{\theta}_1^2 + \frac{1}{2} m_2 \dot{L}_2^{*2} - \frac{m_1}{2} g L_1 - m_2 g L_1$$

Starting with τ_1: ($q_1 = \theta_1^*$)

We look at all parts of the equation with $\dot{\theta}_1$ in it (the rest are constants, and will be 0 when we derivate)

$$\frac{\partial L}{\partial \dot{q}_1} = 2 \cdot \frac{1}{2} I_{1,zz} \dot{\theta}_1 + 2 \cdot \frac{1}{2} m_2 L_2^{*2} \dot{\theta}_1$$

$$= I_{1,zz} \dot{\theta}_1 + m_2 L_2^{*2} \dot{\theta}_1$$

When we derivate with respect to time we must look at all variables in the expression

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} = I_{1,zz} \ddot{\theta}_1 + m_2 L_2^{*2} \ddot{\theta}_1 + 2 m_2 L_2^* \dot{L}_2^* \dot{\theta}_1$$

$$\frac{\partial L}{\partial q_1} = 0$$

Product rule

$u = I_{1,zz}$ $\quad u' = 0$

$v = \ddot{\theta}_1$ $\quad v' = \dddot{\theta}_1$

$\Rightarrow I_{1,zz} \ddot{\theta}_1 + 0 \cdot \dddot{\theta}_1$

When we derivate with respect to time we must both use the Product rule and the Chain rule

$u = m_2 L_2^{*2}$ $\quad u' = 2 m_2 L_2^* \dot{L}_2^*$

$v = \dot{\theta}_1$ $\quad v' = \ddot{\theta}_1$

$\Rightarrow m_2 L_2^{*2} \ddot{\theta}_1 + 2 m_2 L_2^* \dot{L}_2^* \dot{\theta}_1$

for $m_2 L_2^{*2}$: $u = L_2^*$

$(m_2 L_2^{*2})' \cdot u' = 2 m_2 L_2^* \cdot \dot{L}_2^*$

Then Calculating τ_2: $(q_2 = L_2^*)$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_2} = 2 \cdot \frac{1}{2} m_2 \dot{L}_2^* = m_2 \dot{L}_2^*$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2} = m_2 \ddot{L}_2^*$$

$$\frac{\partial \mathcal{L}}{\partial q_2} = 2 \cdot \frac{1}{2} m_2 L_2^* \dot{\theta}_1^2 = m_2 L_2^* \dot{\theta}_1^2$$

Assemble

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) - \frac{\partial \mathcal{L}}{\partial q_1} \\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right) - \frac{\partial \mathcal{L}}{\partial q_2} \end{bmatrix}$$

$$= \begin{bmatrix} I_{1,zz}\ddot{\theta}_1 + m_2 L_2^{*2}\ddot{\theta}_1 + 2 m_2 L_2^* \dot{L}_2^* \dot{\theta}_1 - 0 \\ m_2 \ddot{L}_2^* - m_2 L_2^* \dot{\theta}_1^2 \end{bmatrix}$$

Inertia/torque — acceleration — coreolis/centrifugal forces — velocity — gravity

$$\underline{\tau} = \underbrace{\begin{bmatrix} I_{1,zz} + m_2 L_2^{*2} & 0 \\ 0 & m_2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{L}_2 \end{bmatrix}}_{\ddot{q}} + \underbrace{\begin{bmatrix} m_2 L_2^* \dot{L}_2^* & m_2 L_2^* \dot{\theta}_1 \\ -m_2 L_2^* & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \dot{\theta}_1 \\ \dot{L}_2 \end{bmatrix}}_{\dot{q}} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{g}$$

Other approach

- not by hand (for computing)

$$K = \frac{1}{2}\dot{q}^T \left[\sum_{i=1}^{n} \left\{ m_i J_{v_i}(q)^T J_{v_i}(q) + R_i(q) I_i R_i(q)^T J_{w_i}(q) \right\} \right] \dot{q}$$

$$= \frac{1}{2}\dot{q}^T D(q) \dot{q}$$

$D(q)$ → the inertial matrix

Christoffel symbols

$$C_{ijk} = \frac{1}{2}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right)$$

P → potential energy

$$g_k = \frac{\partial P}{\partial q_k}$$