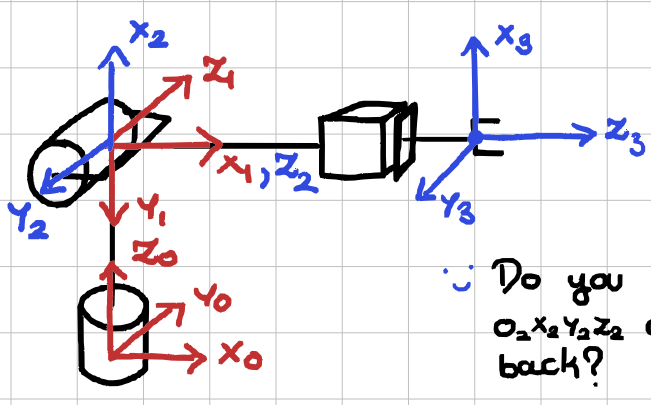
Inverse Kinematics

PURPOSE: FINDING THE JOINT VARIABLES IN TERMS OF THE TOOL'S POSITION AND ORIENTATION. THERE ARE TWO APPROACHES: ANALYTICAL AND GEOMETRIC.

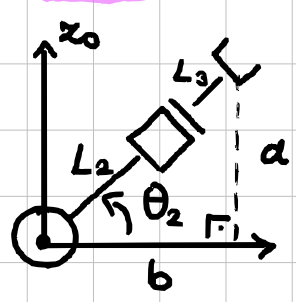
The geometric approach



Do you see why $o_2x_2y_2z_2$ was moved back?

- We project the manipulator onto different planes. We use the base frame often and presume the tool tip coordinates to be (x, y, z). TRIGONOMETRIC IDENTITIES ARE USEFUL.

SIDE VIEW



$z - L_1 = a$

Why? The side view does not show the total height relative to the base.

$b = \pm\sqrt{x^2 + y^2}$

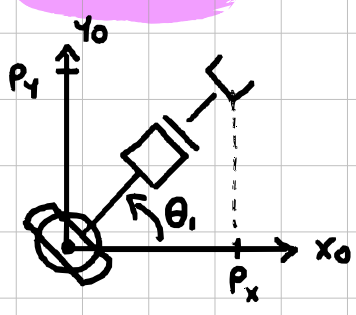
$\theta_2 = \text{atan2}(a, \pm b)$

The hypotenuse is $L_2 + L_3 + d_1^*$

$a^2 + b^2 = (L_2 + L_3 + d_1^*)^2$

$\sqrt{a^2 + b^2} - L_2 - L_3 = d_1^*$

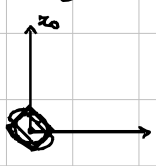
TOP VIEW



$\theta_1 = \text{atan2}\left(\frac{y}{x}\right)$

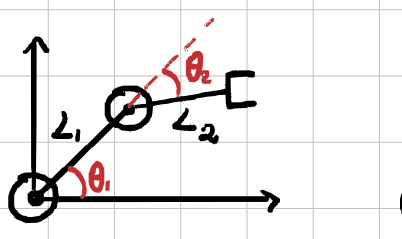
! The total length is not $L_2 + L_3 + d_1^*$ hee. It depends on θ_2.

If the manipulator stands upright, we get the following top plane profile

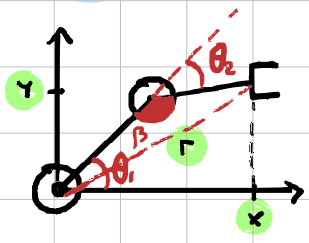


ELBOW UP & DOWN: TWO POSSIBLE CONFIGURATIONS THAT GIVE DIFFERENT ANSWERS

- This example is taken from last year's notes :^)



LOOKING AT ELBOW UP





$\pi = \theta_2 + \beta$

$\cos(\beta) = \cos(\pi - \theta_2) = -\cos(\theta_2)$

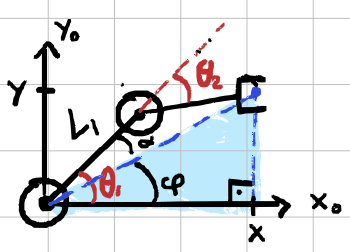
$r^2 = x^2 + y^2$

$r^2 = L_1^2 + L_2^2 - 2L_1L_2\cos\beta$

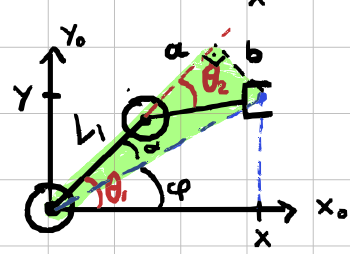
$-\frac{x^2 + y^2 - L_1^2 - L_2^2}{2L_1L_2} = \cos\beta$

$\frac{x^2 + y^2 - L_1^2 - L_2^2}{2L_1L_2} = \cos\theta_2$: D

$\theta_2 = \text{atan2}(\pm\sqrt{1 - D^2}, D)$



- $\theta_1 = \alpha + \varphi$
- $\varphi = \text{atan2}(y, x)$



- $\cos\theta_2 = \frac{a}{L_2}$ $\quad a = L_2\cos\theta_2$
- $\sin\theta_2 = \frac{b}{L_2}$ $\quad b = L_2\sin\theta_2$
- $\alpha = \text{atan2}(L_2\sin\theta_2, L_1 + L_2\cos\theta_2)$
- $\theta_1 = \text{atan2}(y, x) + \text{atan2}(L_2\sin\theta_2, L_1 + L_2\cos\theta_2)$

THE ANALYTICAL APPROACH

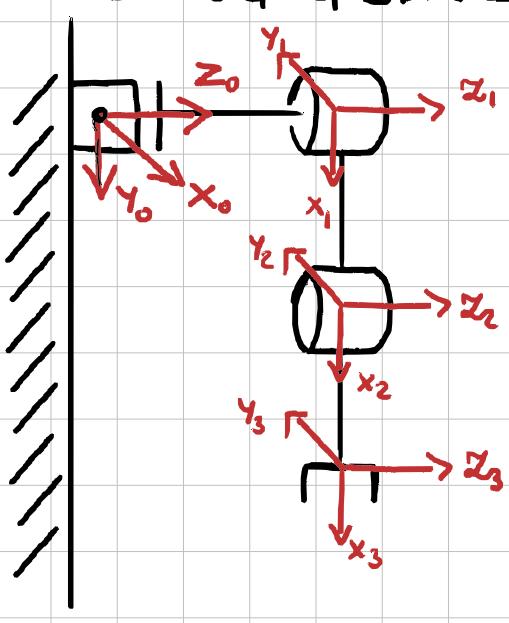
THE LAST COLUMN OF THE FORWARD KINEMATICS MATRIX T CONTAINS THE 3D POSITION VECTOR OF THE TOOL TIP "(a,b,c)". WE SET UP A SYSTEM OF EQUATIONS AND EQUATE THE KNOWN 3D SPATIAL COORDINATES (x,y,z) TO THE EXPRESSIONS (a,b,c) OF THE FORWARD KINEMATICS MATRIX.

$x = a$
$y = b$
$z = c$

a, b, c CONTAIN THE JOINT VARIABLES d_i AND θ_i

$$T = \begin{bmatrix} R & \begin{matrix} a \\ b \\ c \end{matrix} \\ \begin{matrix} \rightarrow \\ 0 \end{matrix} & 1 \end{bmatrix}$$

A CHALLENGING EXAMPLE (FROM 2023):



LINK	a_i	α_i	d_i	θ_i
1	0	0	$L_1 + d_1^*$	$\pi/2$
2	L_2	0	0	θ_2^*
3	L_3	0	0	θ_3^*

♥ TRY FINDING THE FORWARD KINEMATICS MATRICES YOURSELF :)

(CHECK GITHUB FOR THE SOLUTION)

INVERSE KINEMATICS:

$$x = -L_3 s_{23} - L_2 s_2$$
$$y = L_3 c_{23} + L_2 c_2$$
$$z = L_1 + d_1^*$$
$$\hookrightarrow \quad z - L_1 = d_1^*$$

TRY SQUARING x AND y TO SEE IF ANYTHING CANCELS OUT:

$$x^2 = L_3^2 s_{23}^2 + L_2^2 s_2^2 + 2 L_3 L_2 s_{23} s_2$$
$$y^2 = L_3^2 c_{23}^2 + L_2^2 c_2^2 + 2 L_3 L_2 c_{23} c_2$$

$$x^2+y^2 = L_3^2 (s_{23}^2 + c_{23}^2) + L_2^2 (s_2^2 + c_2^2) + 2 L_3 L_2 (s_{23} s_2 + c_{23} c_2)$$

($s_{23}^2 + c_{23}^2 = 1$, $s_2^2 + c_2^2 = 1$)

$$x^2+y^2 = L_3^2 + L_2^2 + 2 L_2 L_3 (s_2^2 c_3 + s_2 s_3 c_3 + c_2^2 c_3 - s_2 s_3 c_3)$$

$$c_3 (s_2^2 + c_2^2) = c_3 \cdot 1$$

$$x^2+y^2 = L_3^2 + L_2^2 + 2 L_2 L_3 c_3$$

$$\frac{x^2+y^2 - L_2^2 - L_3^2}{2 L_2 L_3} = c_3 := q$$

$$\theta_3 = \operatorname{atan2}(\sqrt{1-q^2}, q)$$

NOW, FOR θ_2, USE θ_3 AS A KNOWN CONSTANT

$$y = L_3 c_{23} + L_2 c_2 = L_3 c_2 c_3 - L_3 s_2 s_3 + L_2 c_2$$
$$= c_2 (L_3 c_3 + L_2) - (L_3 s_3) s_2$$

y → c; $(L_3 c_3 + L_2)$ → coefficient a; $(L_3 s_3)$ → coefficient b

USE THE FORMULA

$$a \cos\theta + b \sin\theta = c \iff \operatorname{atan2}\left(\frac{b}{a}\right) \pm \operatorname{atan2}\left(\frac{\sqrt{a^2+b^2+c^2}}{c}\right)$$

$$\theta_2 = \operatorname{atan2}\left(\frac{b}{a}\right) \pm \operatorname{atan2}\left(\frac{\sqrt{a^2+b^2+c^2}}{c}\right)$$

SINGULARITIES

- THE ROBOT LOSES ONE OR MORE DOF
- THE ROBOT CANNOT MOVE IN CERTAIN DIRECTIONS
- MECHANICAL STRAIN ON JOINTS MAY LEAD TO DAMAGE
- WILL BE EXPLAINED FURTHER IN VELOCITY KINEMATICS

