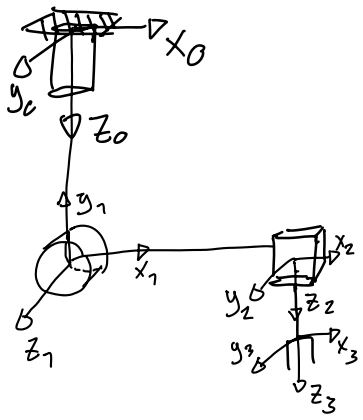
From exam 2019



Velocity Kin.

Task: Find an expression for the Jacobian

Formulas

$$\xi = J\dot{q}$$

$$J = \begin{bmatrix} J_v \\ J_w \end{bmatrix}$$

$$J_{v_i} = \begin{cases} z_{i-1} \times (o_n - o_{i-1}) & \text{revolut} \\ z_{i-1} & \text{prismatic} \end{cases}$$

$$J_{w_i} = \begin{cases} z_{i-1} & \text{revolut} \\ 0 & \text{prismatic} \end{cases}$$

- With a revolut joint, the link will move in both a linear and an angular direction
- With a prismatic joint, the link will only move with a linear motion

$$\vec{a} \times \vec{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

Forward (This is given)

$$A_1 = T_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = T_2^1 = \begin{bmatrix} c_2 & 0 & s_2 & L_2 c_2 \\ s_2 & 0 & -c_2 & L_2 s_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = T_3^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_3^* \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^0 = \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & L_2 c_1 c_2 \\ c_2 s_1 & c_1 & s_1 s_2 & L_2 c_2 s_1 \\ -s_2 & 0 & c_2 & -L_2 s_2 + L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & c_1 s_2 L_3^* + L_2 c_1 c_2 \\ c_2 s_1 & c_1 & s_1 s_2 & s_1 s_2 L_3^* + L_2 c_2 s_1 \\ -s_2 & 0 & c_2 & c_2 L_3^* - L_2 s_2 + L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$z_0^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad o_0^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

everything is related to o_0

$$J = \begin{bmatrix} z_0 \times (o_3 - o_0) & z_1 \times (o_3 - o_1) & z_2 \\ z_0 & z_1 & 0 \end{bmatrix}$$

$$J_{v_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} c_1 s_2 L_3^* + L_2 c_1 c_2 \\ s_1 s_2 L_3^* + L_2 c_2 s_1 \\ c_2 L_3^* - L_2 s_2 + L_1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} c_1 (s_2 L_3^* + c_2 L_2) \\ s_1 (s_2 L_3^* + c_2 L_2) \\ c_2 L_3^* - L_2 s_2 + L_1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -s_1 (s_2 L_3^* + c_2 L_2) \\ c_1 (s_2 L_3^* + c_2 L_2) \\ 0 \end{bmatrix}$$

$$J_{v_2} = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} c_1 (s_2 L_3^* + c_2 L_2) \\ s_1 (s_2 L_3^* + c_2 L_2) \\ c_2 L_3^* - L_2 s_2 + L_1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix}$$

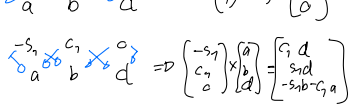
$$= \begin{bmatrix} c_1 (c_2 L_3^* - s_2 L_2) \\ s_1 (c_2 L_3^* - s_2 L_2) \\ -s_1^2 (s_2 L_3^* + c_2 L_2) - c_1^2 (s_2 L_3^* + c_2 L_2) \end{bmatrix}$$

$$= \begin{bmatrix} c_1 (c_2 L_3 - s_2 L_2) \\ s_1 (c_2 L_3 - s_2 L_2) \\ -s_2 L_3 - c_2 L_2 \end{bmatrix}$$

$$J_{v_3} = z_2 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix}$$

$$J = \begin{bmatrix} J_{v_1} & J_{v_2} & J_{v_3} \\ J_{w_1} & J_{w_2} & J_{w_3} \end{bmatrix} = \begin{bmatrix} -s_1 (s_2 L_3^* + c_2 L_2) & c_1 (c_2 L_3 - s_2 L_2) & c_1 s_2 \\ c_1 (s_2 L_3^* + c_2 L_2) & s_1 (c_2 L_3 - s_2 L_2) & s_1 s_2 \\ 0 & -s_2 L_3 - c_2 L_2 & c_2 \\ 0 & -s_2 & 0 \\ 0 & c_2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Calculations:



Alternative approach using Partial derivative

$$\vec{O}_3^0 = \begin{bmatrix} c_1(L_2c_2 + L_3^* s_2) \\ s_1(L_2c_2 + L_3^* s_2) \\ L_1 - L_2 s_2 + L_3^* c_2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$J_v = \begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} & \frac{\partial x}{\partial L_3^*} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} & \frac{\partial y}{\partial L_3^*} \\ \frac{\partial z}{\partial \theta_1} & \frac{\partial z}{\partial \theta_2} & \frac{\partial z}{\partial L_3^*} \end{bmatrix} = \begin{bmatrix} -s_1(L_2c_2 + L_3^* s_2) & c_1(-L_2 s_2 + L_3^* c_2) & c_1 s_2 \\ c_1(L_2c_2 + L_3^* s_2) & s_1(-L_2 s_2 + L_3^* c_2) & s_1 s_2 \\ 0 & -L_2c_2 - L_3^* s_2 & c_2 \end{bmatrix}$$

Example with values

$\theta_1 = 45°$ $\quad L_1 = 3m$

$\theta_2 = 60°$ $\quad L_2 = 2m$

$L_3 = 1m$

$$\dot{q} = \begin{bmatrix} \dot{\theta}_1^* \\ \dot{\theta}_2^* \\ \dot{d}_3^* \end{bmatrix} = \begin{bmatrix} 0{,}2 \text{ rad/s} \\ 0{,}7 \text{ rad/s} \\ 0{,}005 \text{ m/s} \end{bmatrix}$$

dot means derivative of the position
So $\dot{q}$ is a matrix with the velocities of each joint ($V(t) = S'(t)$, where $S(t)$ is position)

Task Find the linear and angular velocity of the tooltip relative to baseframe:

$$J\dot{q} = \begin{bmatrix} -s_1(s_2L_3^* + c_2L_2) & c_1(c_2L_3 - s_2L_2) & c_1s_2 \\ c_1(s_2L_3^* + c_2L_2) & s_1(c_2L_3 - s_2L_2) & s_1s_2 \\ 0 & -s_2L_3 - c_2L_2 & c_2 \\ 0 & -s_2 & 0 \\ 0 & c_2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^* \\ \dot{\theta}_2^* \\ \dot{d}_3^* \end{bmatrix} = \begin{bmatrix} -1{,}372 & -0{,}871 & 0{,}612 \\ 1{,}372 & -0{,}1871 & 0{,}612 \\ 0 & -1{,}866 & 0{,}5 \\ 0 & -0{,}866 & 0 \\ 0 & 0{,}5 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0{,}2 \text{ rad/s} \\ 0{,}7 \text{ rad/s} \\ 0{,}005 \text{ m/s} \end{bmatrix} = \begin{bmatrix} -0{,}869 \\ -0{,}1344 \\ -1{,}304 \\ -0{,}609 \\ 0{,}135 \\ 0{,}12 \end{bmatrix}$$

$$J_w q = \begin{bmatrix} -s_2\dot{\theta}_2 \\ c_2\dot{\theta}_2 \\ \dot{\theta}_1 \end{bmatrix}$$ → depends on velocity of θ_2; → depends on velocity of θ_1

The Same applies to J_v,
So the Velocity in x-direction depends on θ_1^*, θ_2^* and d_3^*
the Velocity in y-direction depends on θ_1^*, θ_2^* and d_3^*
the Velocity in z-direction depends on θ_2^* and d_3^*

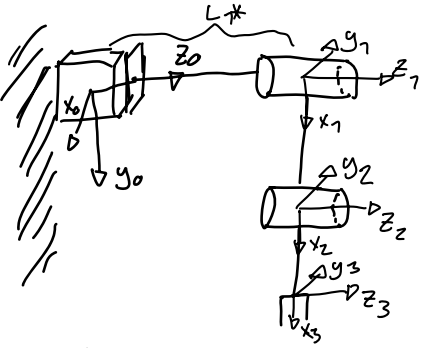
Linear Velocity

We usually look for the velocity in the x-, y- and z direction

Angular Velocity

The velocity of rotation around the x-, y- and z-axis

From exam 2023:



DH	θ	α	d	a
1	90°	0	L_1^*	0
2	θ_2^*	0	0	L_2
3	θ_3^*	0	0	L_3

forward (given)

$$T_1^0 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^0 = \begin{bmatrix} -S_2 & -C_2 & 0 & -L_2 S_2 \\ C_2 & -S_2 & 0 & L_2 C_2 \\ 0 & 0 & 1 & L_1^* \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = \begin{bmatrix} -S_{23} & -C_{23} & 0 & -L_3 S_{23} - L_2 S_2 \\ C_{23} & -S_{23} & 0 & L_3 C_{23} + L_2 C_2 \\ 0 & 0 & 1 & L_1^* \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$J_v = \begin{bmatrix} \frac{\partial x}{\partial d_1^*} & \frac{\partial x}{\partial \theta_2} & \frac{\partial x}{\partial \theta_3} \\ \frac{\partial y}{\partial d_1^*} & \frac{\partial y}{\partial \theta_2} & \frac{\partial y}{\partial \theta_3} \\ \frac{\partial z}{\partial d_1^*} & \frac{\partial z}{\partial \theta_2} & \frac{\partial z}{\partial \theta_3} \end{bmatrix}$$

$$J_v = \begin{bmatrix} 0 & -L_3 C_{23} - L_2 C_2 & -L_3 C_{23} \\ 0 & -L_3 S_{23} - L_2 S_2 & -L_3 S_{23} \\ 1 & 0 & 0 \end{bmatrix}$$

$$J_w = \begin{bmatrix} 0 & z_1^0 & z_2^0 \end{bmatrix}$$

$$J_w = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Formulas:

$\cos'(\theta) = -\sin(\theta)$

$\sin'(\theta) = \cos(\theta)$

$$\frac{d}{d\theta_a}(\sin_{ab}) = \frac{d}{d\theta_a}(\sin_a \cos_b + \cos_a \sin_b)$$

$$= \cos_a \cos_b - \sin_a \sin_b$$

$$= \cos_{ab}$$

$$\frac{d}{d\theta_a}(\cos_{ab}) = \frac{d}{d\theta_a}(\cos_a \cos_b - \sin_a \sin_b)$$

$$= -\sin_a \cos_b - \cos_a \sin_b$$

$$= -1(\sin_a \cos_b + \cos_a \sin_b)$$

$$= -\sin_{ab}$$

Find Singularities

$$\det|J_v| = \begin{vmatrix} 0 & -l_2c_2 - l_3c_{23} & -l_3c_{23} \\ 0 & -l_2s_2 - l_3s_{23} & -l_3s_{23} \\ 1 & 0 & 0 \end{vmatrix} = 0$$

Formula

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= +a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - d\begin{vmatrix} b & c \\ h & i \end{vmatrix} + g\begin{vmatrix} b & c \\ e & f \end{vmatrix}$$

$$= a(ei-fh) - d(bi-ch) + g(bf-ce)$$

$$= +a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei-hf) - b(di-gf) + c(dh-eg)$$

$$0\begin{vmatrix} -l_2s_2 - l_3s_{23} & -l_3s_{23} \\ 0 & 0 \end{vmatrix} - 0\begin{vmatrix} -l_2c_2 - l_3c_{23} & -l_3c_{23} \\ 0 & 0 \end{vmatrix} + 1\begin{vmatrix} -l_2c_2 - l_3c_{23} & -l_3c_{23} \\ -l_2s_2 - l_3s_{23} & -l_3s_{23} \end{vmatrix} = 0$$

$$(-l_2c_2 - l_3c_{23})(-l_3s_{23}) - (-l_3c_{23})(-l_2s_2 - l_3s_{23}) = 0$$

$$l_2l_3c_2s_{23} + l_3^2c_{23}s_{23} - l_2l_3s_2c_{23} - l_3^2s_{23}c_{23} = 0$$

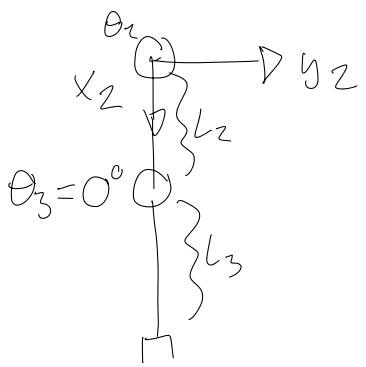
$$l_2l_3(c_2s_{23} - s_2c_{23}) = 0$$

$$-l_2l_3(s_2c_{23} - c_2s_{23}) = -l_2l_3s_{2-(2+3)} = -l_2l_3s_{-3}$$

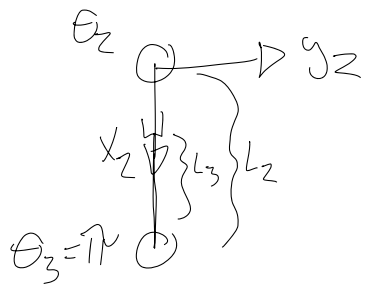
$$= l_2l_3s_3 = 0$$

$$s_3 = 0 \Rightarrow \underline{\theta_3 = 0} \vee \underline{\theta_3 = \pm\pi}$$

Draw Singularities



Workspace boundry



over itself