

# Introduction to Robotics (Fag 3480)

Vår 2010

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RIKSHOSPITALET

HELSE SØR-ØST

# Ch. 2: Rigid Body Motions and Homogeneous Transforms

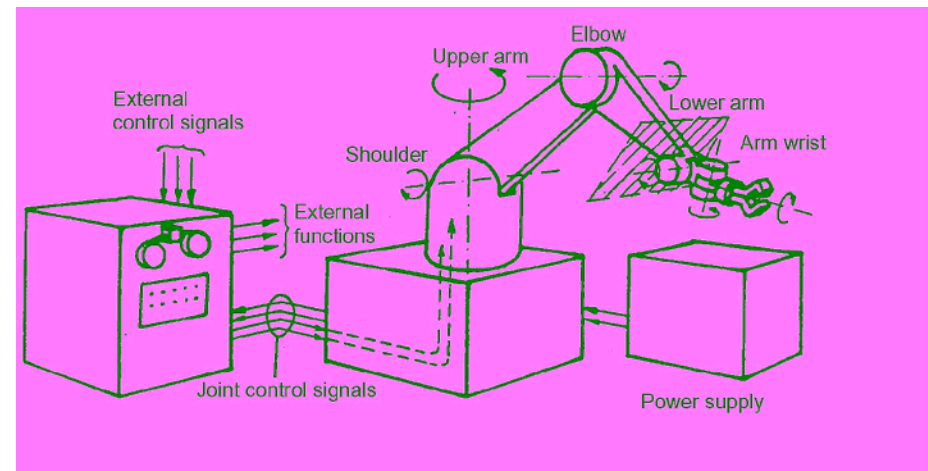
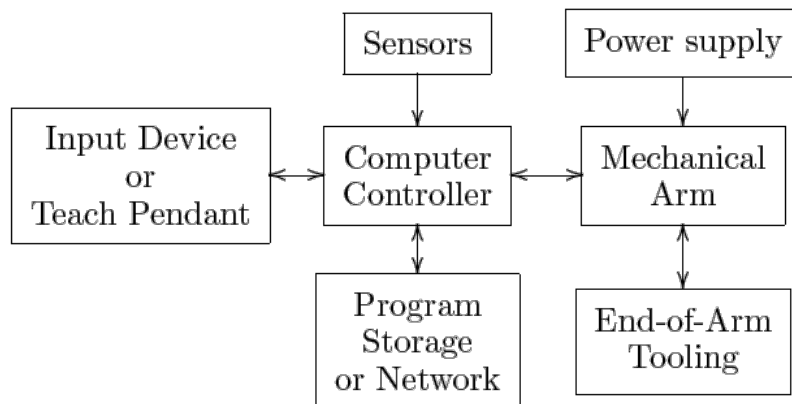
side 2

# Industrial robots

High precision and repetitive tasks

Pick and place, painting, etc

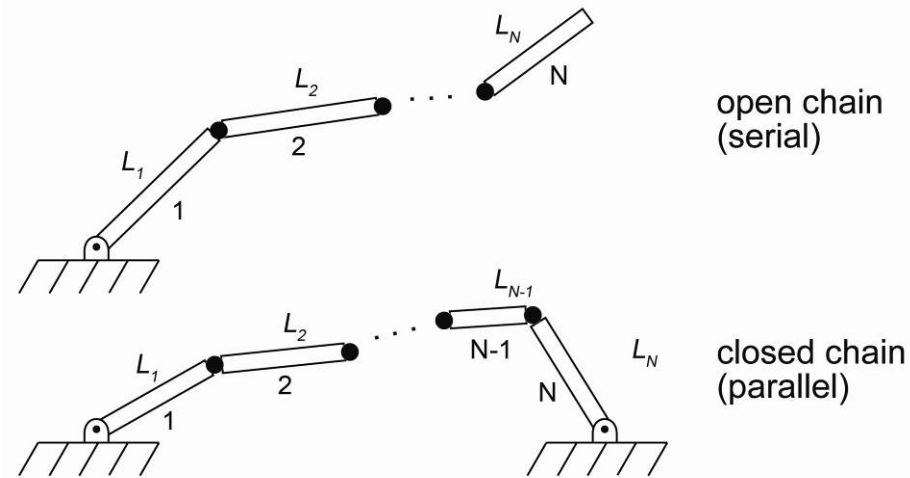
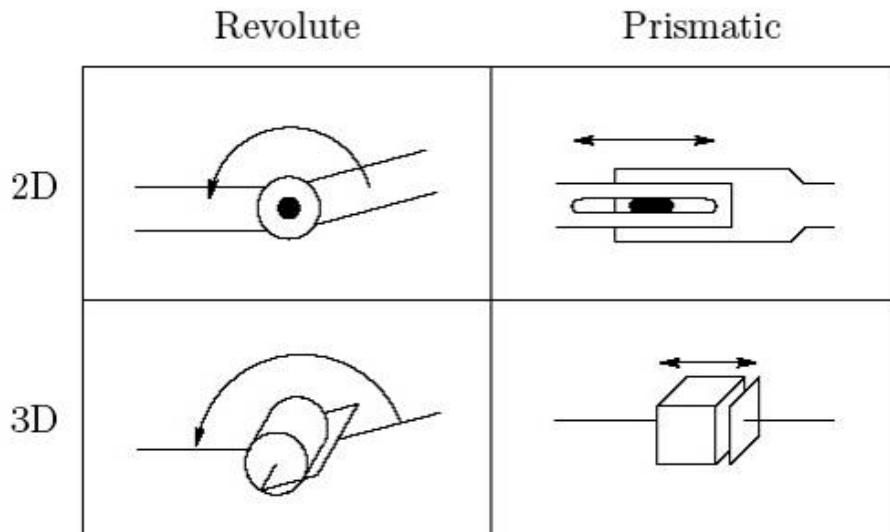
Hazardous environments



# Representations

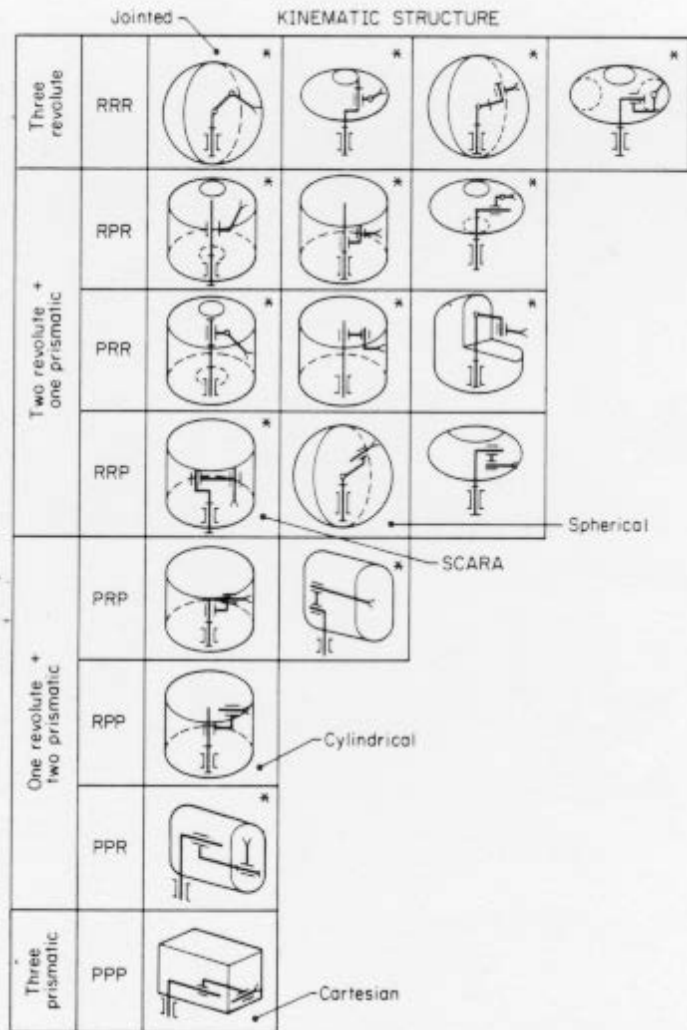
For the majority of this class, we will consider robotic manipulators as open or closed chains of links and joints

Two types of joints: revolute ( $\theta$ ) and prismatic ( $d$ )



side 4

# Arm configurations



The most frequent arm configurations are :

- Open kinematic chains :

- Jointed articulated or anthropomorphic (human-like arms) (RRR)
- Spherical (RRP)
- Scara (RRP)
- Cylindrical (RPP)
- Cartesian (PPP)
- Multi-jointed (RRRRRR.....) , Redundant configurations

- Closed kinematic chains

# Definitions

## End-effector/Tool

Device that is in direct contact with the environment. Usually very task-specific

## Configuration

Complete specification of every point on a manipulator

set of all possible configurations is the *configuration space*

For rigid links, it is sufficient to specify the configuration space by the joint angles,  $q = [q_1 \quad q_2 \quad \dots \quad q_n]^T$

## State space

Current configuration (joint positions  $q$ ) and velocities  $\dot{q}$

## Work space

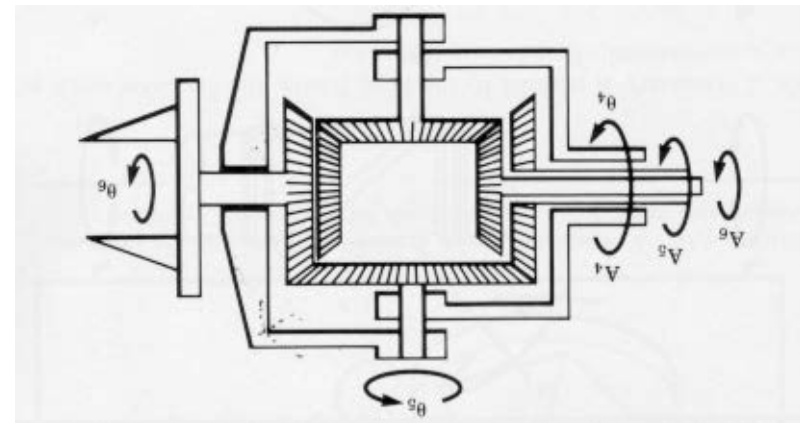
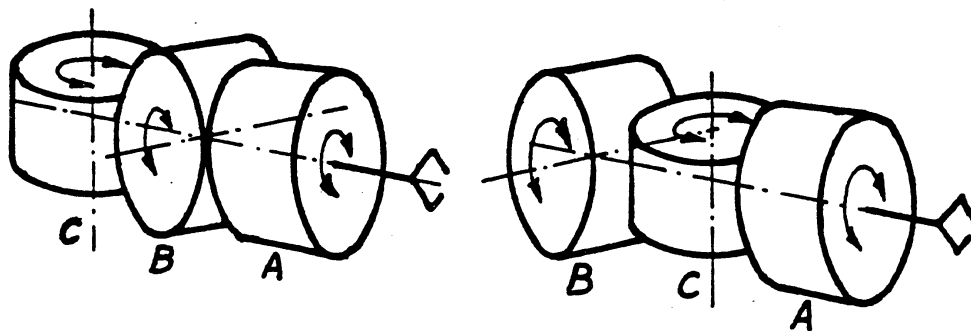
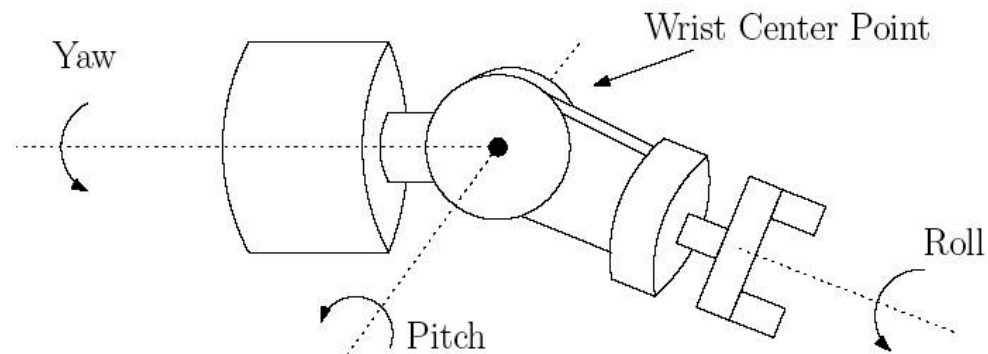
The reachable space the tool can achieve

Reachable workspace

Dextrous workspace

# Common configurations: wrists

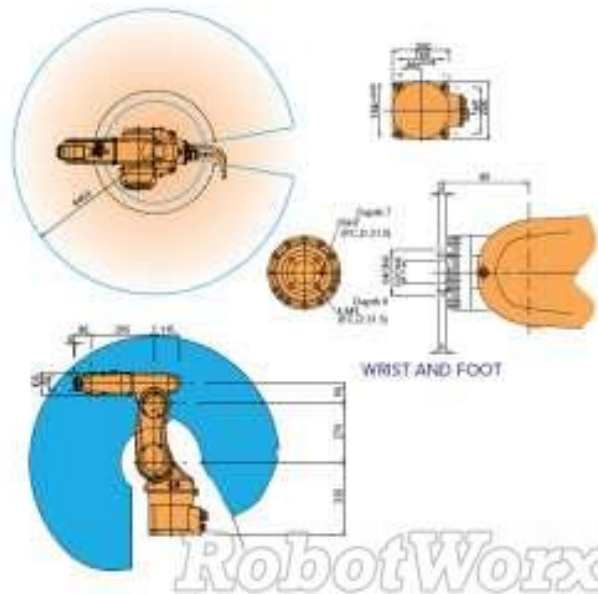
Many manipulators will be a sequential chain of links and joints forming the 'arm' with multiple DOFs concentrated at the 'wrist'



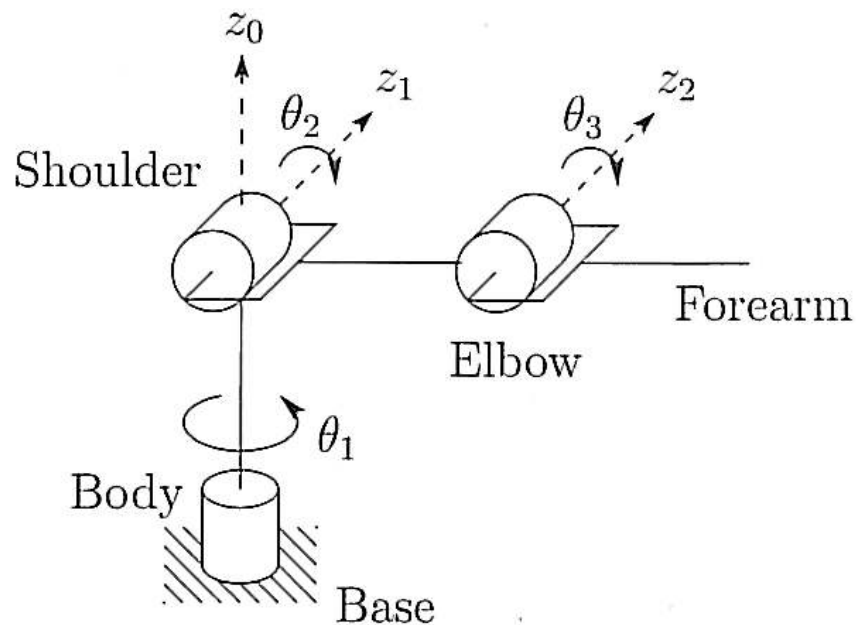
# Common configurations: elbow manipulator

Anthropomorphic arm: ABB IRB1400 or KUKA

Very similar to the lab arm NACHI (RRR)



# Antropomorphic arm (RRR)



# Common configurations: SCARA (RRP)

## Adept Cobra s600/s800 Robot User's Guide

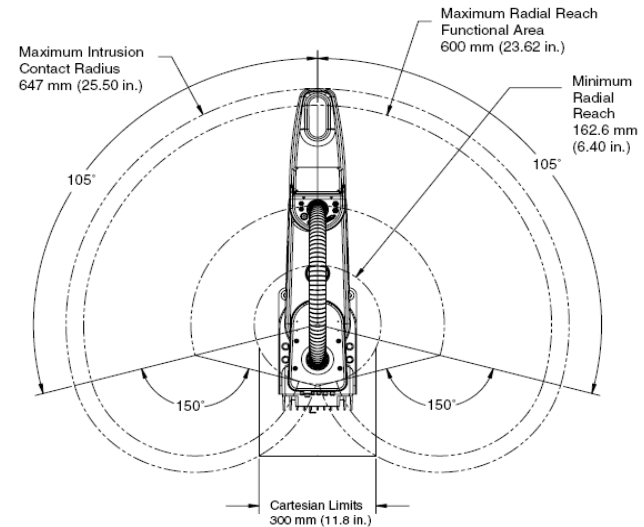
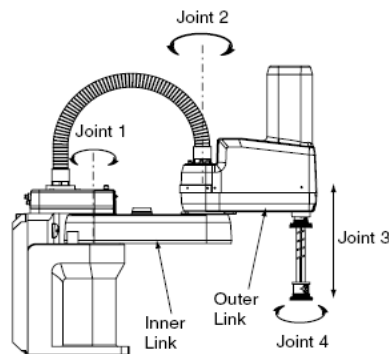


Figure 8-7. Adept Cobra s600 Robot Working Envelope

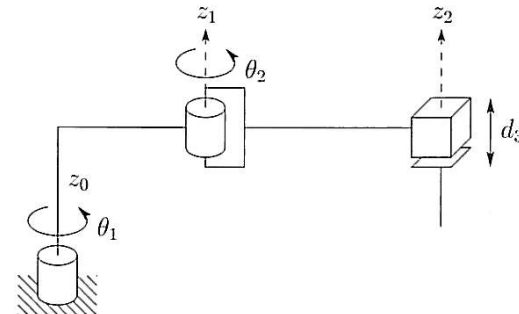
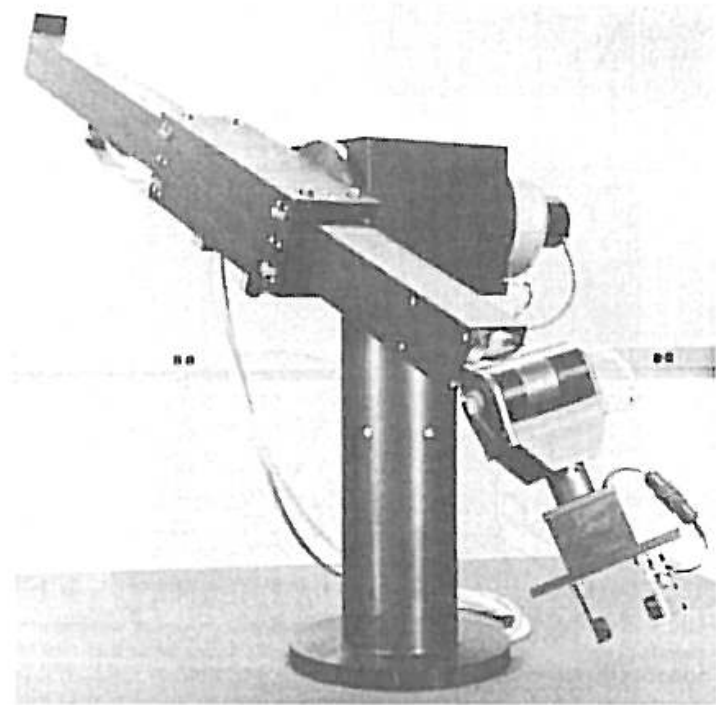
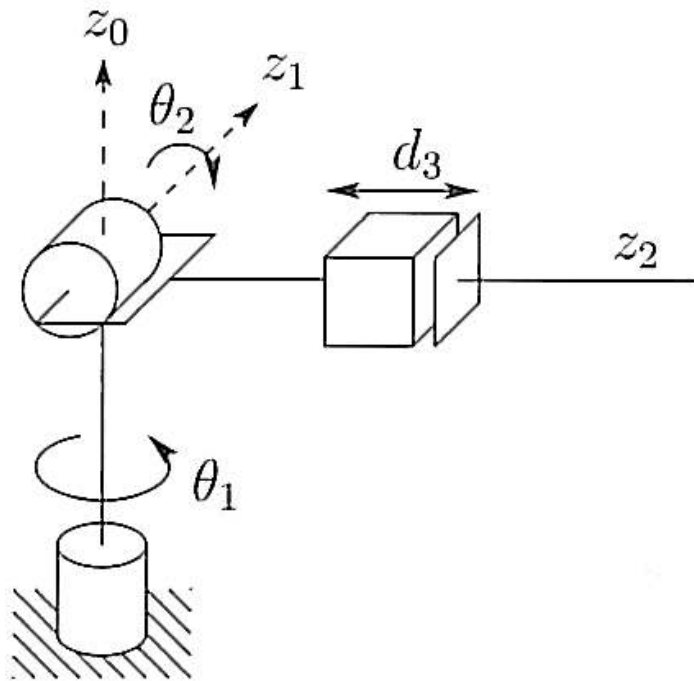


Figure 1.13: Symbolic representation of the SCARA arm.

# Spherical Manipulator (RRP)



# Common configurations: cylindrical robot (RPP)

workspace forms a cylinder

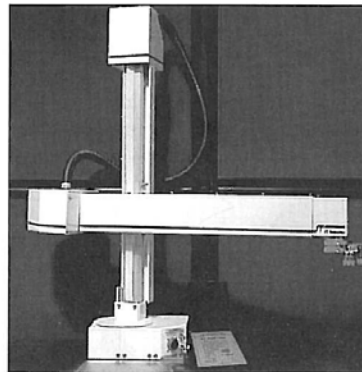
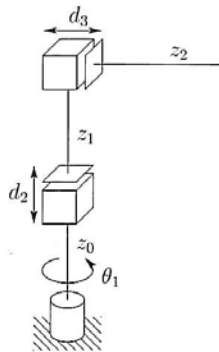


Figure 1.15: The Seiko RT3300 Robot cylindrical robot. Cylindrical robots are often used in materials transfer tasks. (Photo courtesy of Epson Robots.)

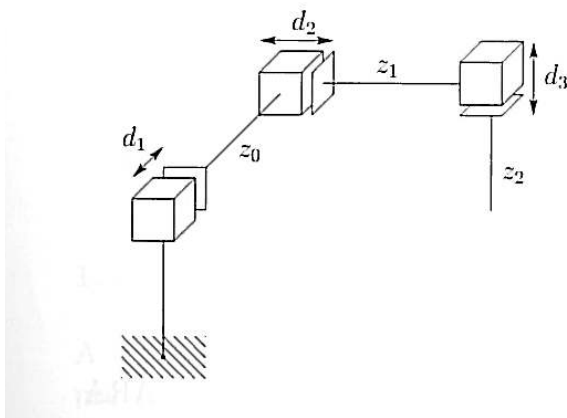


*Seiko RT3300 Robot*

# Common configurations: Cartesian robot (PPP)

Increased structural rigidity, higher precision

Pick and place operations



*Epson Cartesian robot  
(EZ-modules)*

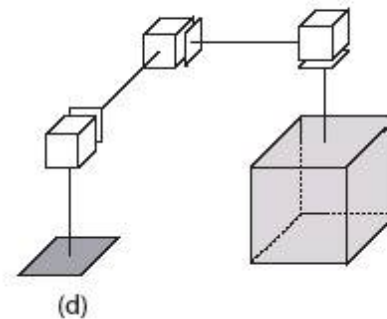
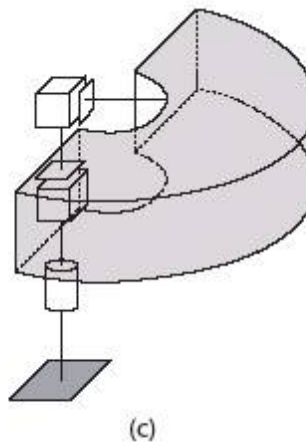
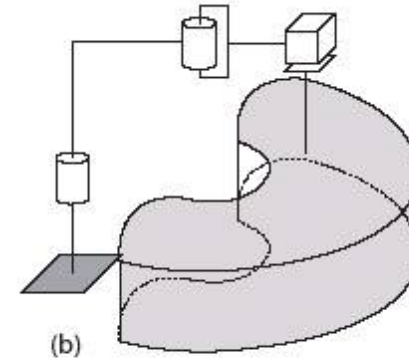
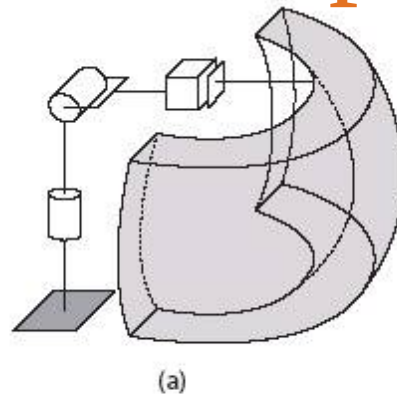
# Workspace comparison

(a) spherical

(b) SCARA

(c) cylindrical

(d) Cartesian



# Parallel manipulators

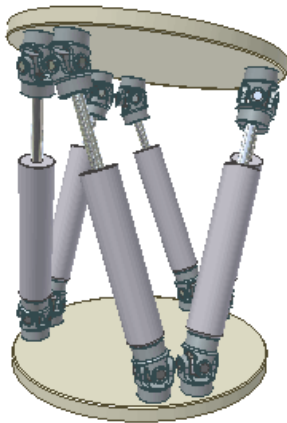
some of the links will form a closed chain with ground

Advantages:

Motors can be proximal: less powerful, higher bandwidth, easier to control

Disadvantages:

Generally less motion, kinematics can be challenging



6DOF Stewart platform

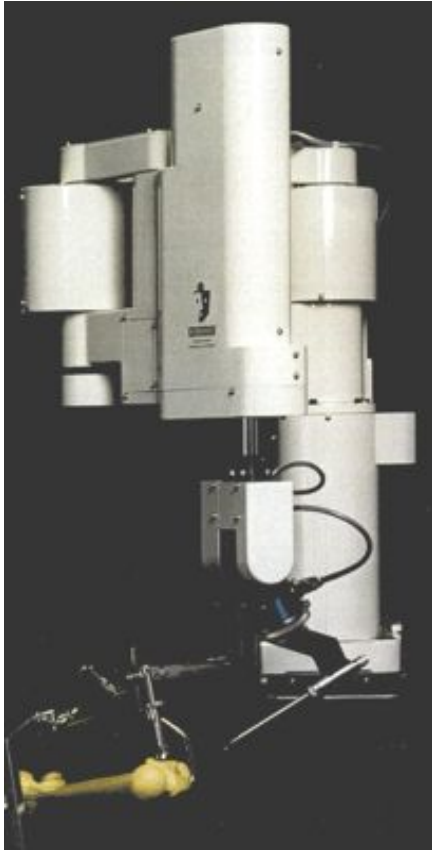
*ABB IRB940 Tricept*



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# Image-guided robots

ROBODOC –  
Integrated Surgical Systems Inc.



PathFinder –  
Armstrong HealthCare Lmt.

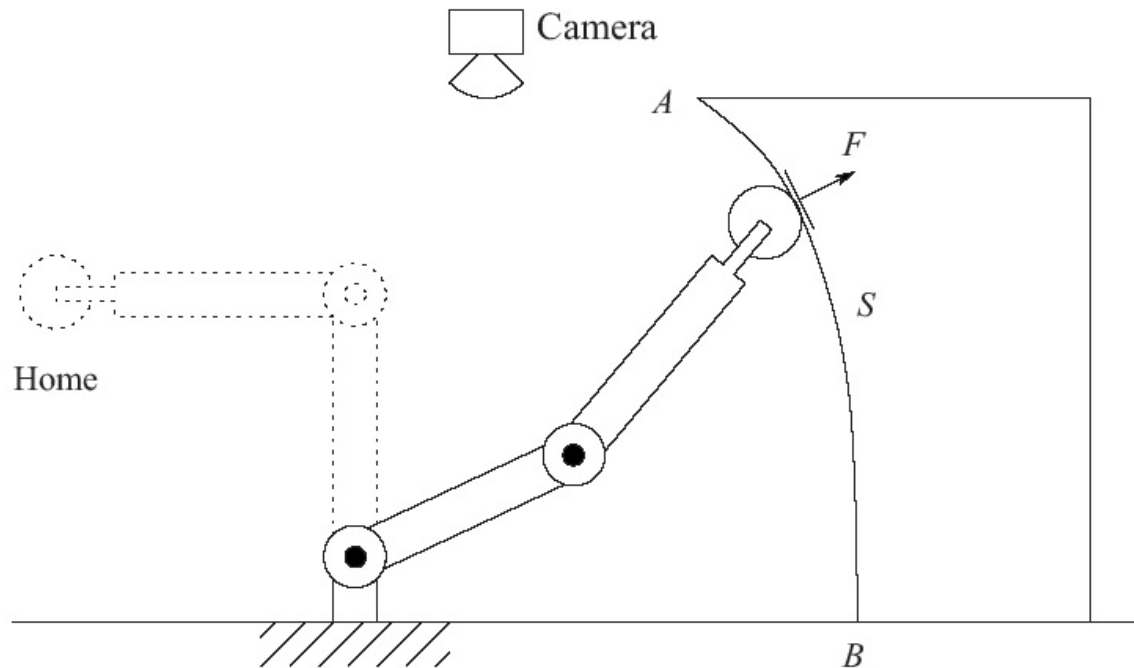


CASPAR - Maquet



# Simple example: control of a 2DOF planar manipulator

Move from 'home' position and follow the path AB with a constant contact force  $F$  all using visual feedback



# Coordinate frames & forward kinematics

Three coordinate frames: (0) (1) (2)

Positions:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} a_1 \cos(\theta_1) \\ a_1 \sin(\theta_1) \end{bmatrix}$$

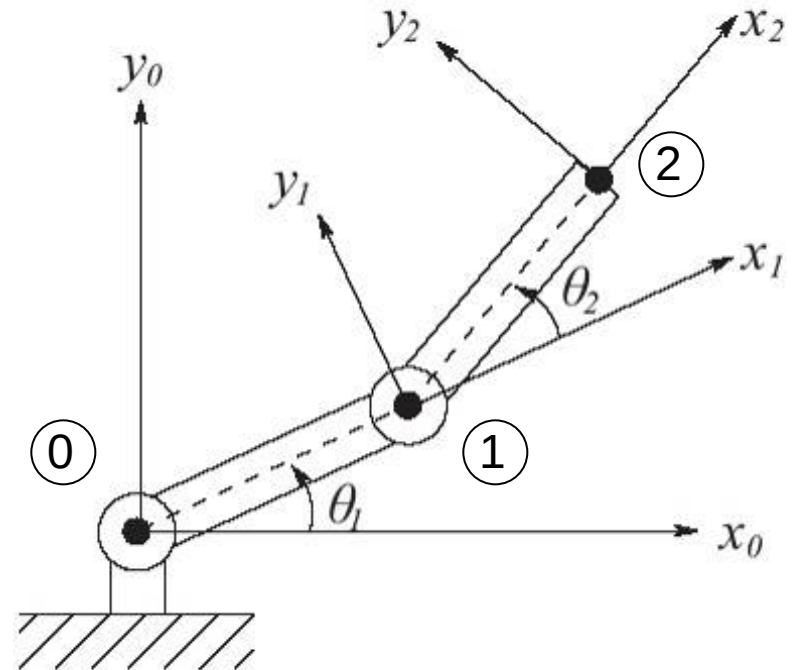
$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_1 \cos(\theta_1) + a_2 \cos(\theta_1 + \theta_2) \\ a_1 \sin(\theta_1) + a_2 \sin(\theta_1 + \theta_2) \end{bmatrix} \equiv \begin{bmatrix} x \\ y \end{bmatrix}_t$$

$$\hat{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{y}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Orientation of the tool frame:

$$\hat{x}_2 = \begin{bmatrix} \cos(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) \end{bmatrix}, \hat{y}_2 = \begin{bmatrix} -\sin(\theta_1 + \theta_2) \\ \cos(\theta_1 + \theta_2) \end{bmatrix}$$

$$R_2^0 = \begin{bmatrix} \hat{x}_2 \cdot \hat{x}_0 & \hat{y}_2 \cdot \hat{x}_0 \\ \hat{x}_2 \cdot \hat{y}_0 & \hat{y}_2 \cdot \hat{y}_0 \end{bmatrix} = \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix}$$



# Ch. 2: Rigid Body Motions and Homogeneous Transforms

# Representing position

Definition: coordinate frame

A set  $n$  of orthonormal basis vectors spanning  $\mathbf{R}^n$

For example,

$$\hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

When representing a point  $p$ , we need to specify a coordinate frame

With respect to  $o_0$ :

$$p^0 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

With respect to  $o_1$ :

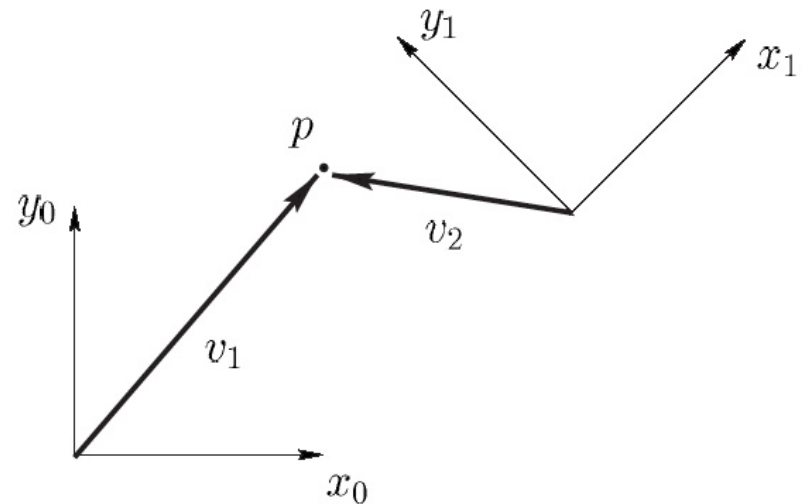
$$p^1 = \begin{bmatrix} -2.8 \\ 4.2 \end{bmatrix}$$

$v_1$  and  $v_2$  are invariant geometric entities

But the representation is dependant

upon choice of coordinate frame

$$v_1^0 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, v_1^1 = \begin{bmatrix} 7.77 \\ 0.8 \end{bmatrix}, v_2^0 = \begin{bmatrix} -5.1 \\ 1 \end{bmatrix}, v_2^1 = \begin{bmatrix} -2.8 \\ 4.2 \end{bmatrix}$$



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# Rotations

## 2D rotations

Representing one coordinate frame in terms of another

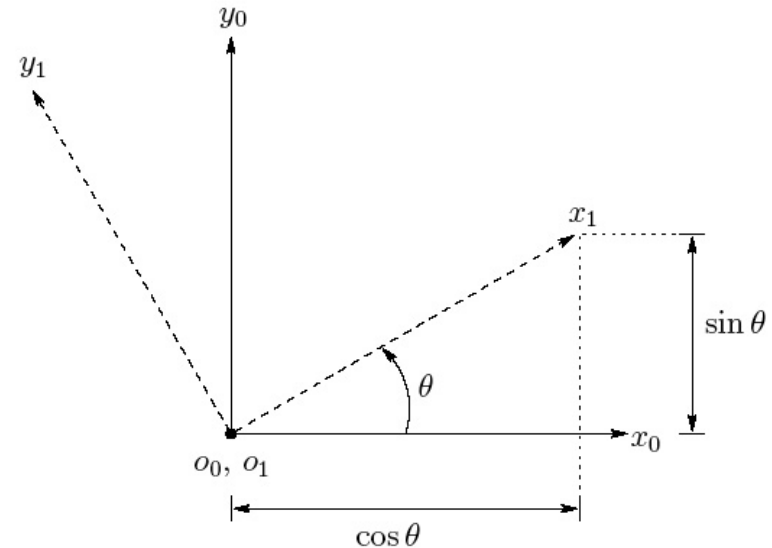
$$R_1^0 = \begin{bmatrix} x_1^0 & y_1^0 \end{bmatrix}$$

Where the unit vectors are defined as:

$$x_1^0 = \|\hat{x}_0\| \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}, y_1^0 = \|\hat{y}_0\| \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

This is a **rotation matrix**



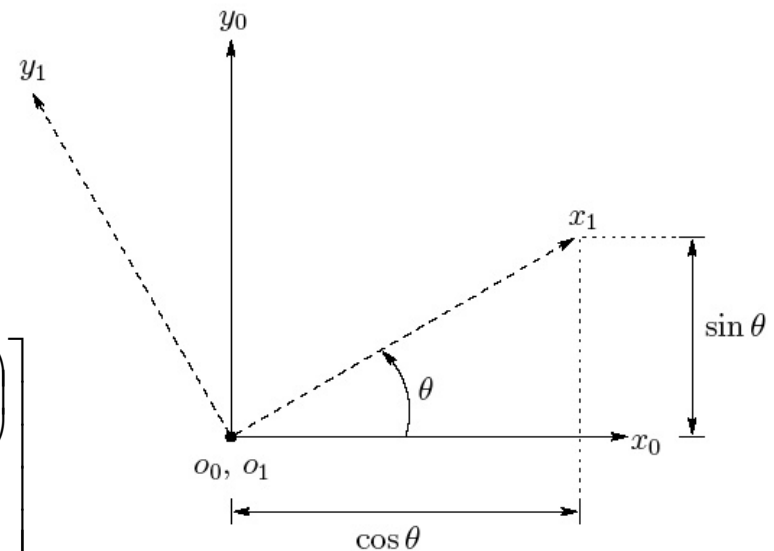
# Alternate approach

## Rotation matrices as projections

Projecting the axes of from  $o_1$  onto the axes of frame  $o_0$

$$x_1^0 = \begin{bmatrix} \hat{x}_1 \cdot \hat{x}_0 \\ \hat{x}_1 \cdot \hat{y}_0 \end{bmatrix}, y_1^0 = \begin{bmatrix} \hat{y}_1 \cdot \hat{x}_0 \\ \hat{y}_1 \cdot \hat{y}_0 \end{bmatrix}$$

$$\begin{aligned} R_1^0 &= \begin{bmatrix} \hat{x}_1 \cdot \hat{x}_0 & \hat{y}_1 \cdot \hat{x}_0 \\ \hat{x}_1 \cdot \hat{y}_0 & \hat{y}_1 \cdot \hat{y}_0 \end{bmatrix} \\ &= \begin{bmatrix} \|\hat{x}_1\| \|\hat{x}_0\| \cos \theta & \|\hat{y}_1\| \|\hat{x}_0\| \cos \left( \theta + \frac{\pi}{2} \right) \\ \|\hat{x}_1\| \|\hat{y}_0\| \cos \left( \frac{\pi}{2} - \theta \right) & \|\hat{y}_1\| \|\hat{y}_0\| \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \end{aligned}$$



# Properties of rotation matrices

Inverse rotations:

$$\begin{aligned} R_0^1 &= \begin{bmatrix} \hat{x}_0 \cdot \hat{x}_1 & \hat{y}_0 \cdot \hat{x}_1 \\ \hat{x}_0 \cdot \hat{y}_1 & \hat{y}_0 \cdot \hat{y}_1 \end{bmatrix} \\ &= \begin{bmatrix} \|\hat{x}_0\| \|\hat{x}_1\| \cos \theta & \|\hat{y}_0\| \|\hat{x}_1\| \cos\left(\frac{\pi}{2} - \theta\right) \\ \|\hat{x}_0\| \|\hat{y}_1\| \cos\left(\theta + \frac{\pi}{2}\right) & \|\hat{y}_0\| \|\hat{y}_1\| \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = (R_1^0)^T \end{aligned}$$

Or, another interpretation uses odd/even properties:

$$\begin{aligned} R_0^1 &= \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = (R_1^0)^T \end{aligned}$$

# Properties of rotation matrices

Inverse of a rotation matrix:

$$\begin{aligned}(R_1^0)^{-1} &= \begin{bmatrix} \hat{x}_1 \cdot \hat{x}_0 & \hat{y}_1 \cdot \hat{x}_0 \\ \hat{x}_1 \cdot \hat{y}_0 & \hat{y}_1 \cdot \hat{y}_0 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \|\hat{x}_1\| \|\hat{x}_0\| \cos \theta & \|\hat{y}_1\| \|\hat{x}_0\| \cos\left(\theta - \frac{\pi}{2}\right) \\ \|\hat{x}_1\| \|\hat{y}_0\| \cos\left(\frac{\pi}{2} - \theta\right) & \|\hat{y}_1\| \|\hat{y}_0\| \cos \theta \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}^{-1} = \frac{1}{\det(R_1^0)} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = (R_1^0)^T\end{aligned}$$

The determinant of a rotation matrix is always  $\pm 1$

+1 if we only use right-handed convention

# Properties of rotation matrices

Summary:

Columns (rows) of  $R$  are mutually orthogonal

Each column (row) of  $R$  is a unit vector

$$R^T = R^{-1}$$

$$\det(R) = 1$$

The set of all  $n \times n$  matrices that have these properties are called the **Special Orthogonal group** of order  $n$

$$R \in SO(n)$$

# 3D rotations

General 3D rotation:

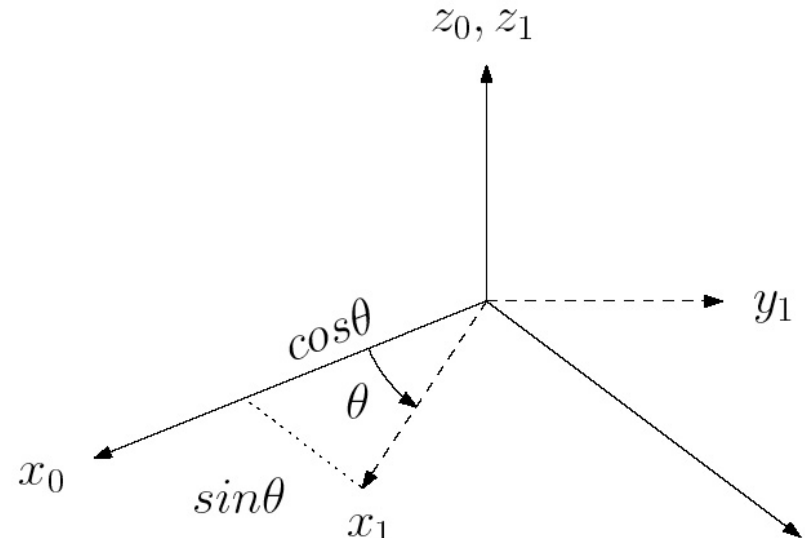
$$R_1^0 = \begin{bmatrix} \hat{x}_1 \cdot \hat{x}_0 & \hat{y}_1 \cdot \hat{x}_0 & \hat{z}_1 \cdot \hat{x}_0 \\ \hat{x}_1 \cdot \hat{y}_0 & \hat{y}_1 \cdot \hat{y}_0 & \hat{z}_1 \cdot \hat{y}_0 \\ \hat{x}_1 \cdot \hat{z}_0 & \hat{y}_1 \cdot \hat{z}_0 & \hat{z}_1 \cdot \hat{z}_0 \end{bmatrix} \in SO(3)$$

Special cases

Basic rotation matrices

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$



$$R_{z,\theta} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Properties of rotation matrices (cont'd)

$SO(3)$  is a group under multiplication

Closure:      if  $R_1, R_2 \in SO(3) \Rightarrow R_1 R_2 \in SO(3)$

Identity:       $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in SO(3)$

Inverse:       $R^T = R^{-1}$

Associativity:       $(R_1 R_2) R_3 = R_1 (R_2 R_3)$

—————> Allows us to combine rotations:

$$R_{ac} = R_{ab} R_{bc}$$

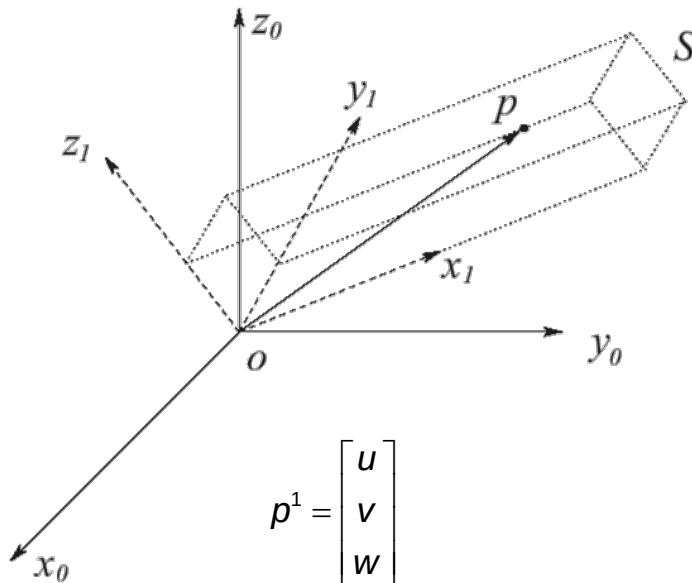
In general, members of  $SO(3)$  do not commute

$$R_1 R_2 \neq R_2 R_1$$

# Rotational transformations

Now assume  $p$  is a fixed point on the rigid object with fixed coordinate frame  $o_1$

The point  $p$  can be represented in the frame  $o_0$  ( $p^0$ ) again by the projection onto the base frame



$$\begin{aligned} p^0 &= \begin{bmatrix} p^1 \cdot \hat{x}_0 \\ p^1 \cdot \hat{y}_0 \\ p^1 \cdot \hat{z}_0 \end{bmatrix} \\ &= \begin{bmatrix} (u\hat{x}_1 + v\hat{y}_1 + w\hat{z}_1) \cdot \hat{x}_0 \\ (u\hat{x}_1 + v\hat{y}_1 + w\hat{z}_1) \cdot \hat{y}_0 \\ (u\hat{x}_1 + v\hat{y}_1 + w\hat{z}_1) \cdot \hat{z}_0 \end{bmatrix} \\ &= \begin{bmatrix} u\hat{x}_1 \cdot \hat{x}_0 + v\hat{y}_1 \cdot \hat{x}_0 + w\hat{z}_1 \cdot \hat{x}_0 \\ u\hat{x}_1 \cdot \hat{y}_0 + v\hat{y}_1 \cdot \hat{y}_0 + w\hat{z}_1 \cdot \hat{y}_0 \\ u\hat{x}_1 \cdot \hat{z}_0 + v\hat{y}_1 \cdot \hat{z}_0 + w\hat{z}_1 \cdot \hat{z}_0 \end{bmatrix} \\ &= \begin{bmatrix} \hat{x}_1 \cdot \hat{x}_0 & \hat{y}_1 \cdot \hat{x}_0 & \hat{z}_1 \cdot \hat{x}_0 \\ \hat{x}_1 \cdot \hat{y}_0 & \hat{y}_1 \cdot \hat{y}_0 & \hat{z}_1 \cdot \hat{y}_0 \\ \hat{x}_1 \cdot \hat{z}_0 & \hat{y}_1 \cdot \hat{z}_0 & \hat{z}_1 \cdot \hat{z}_0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = R_1^0 p^1 \end{aligned}$$

# Rotating a vector

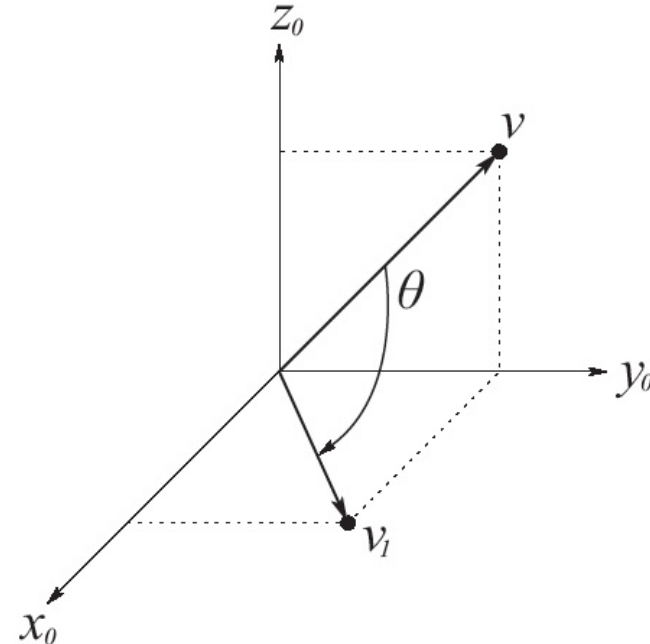
Another interpretation of a rotation matrix:

Rotating a vector about an axis in a fixed frame

Ex: rotate  $v^0$  about  $y_0$  by  $\pi/2$

$$v^0 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} v^1 &= R_{y, \pi/2} v^0 \\ &= \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}_{\theta=\pi/2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$



# Rotation matrix summary

Three interpretations for the role of rotation matrix:

Representing the coordinates of a point in two different frames

Orientation of a transformed coordinate frame with respect to a fixed frame

Rotating vectors in the same coordinate frame

# Similarity transforms

All coordinate frames are defined by a set of basis vectors

These span  $\mathbf{R}^n$

Ex: the unit vectors  $i, j, k$

In linear algebra, a  $n \times n$  matrix  $A$  is a mapping from  $\mathbf{R}^n$  to  $\mathbf{R}^n$

$y = Ax$ , where  $y$  is the image of  $x$  under the transformation  $A$

Think of  $x$  as a linear combination of unit vectors (basis vectors), for example the unit vectors:

$$e_1 = [1 \ 0 \ \dots \ 0]^T, \dots, e_n = [0 \ 0 \ \dots \ 1]^T$$

Then the columns of  $A$  are the images of these basis vectors

If we want to represent vectors with respect to a different basis, e.g.:  $f_1, \dots, f_n$ , the transformation  $A$  can be represented by:

$$A' = T^{-1}AT$$

Where the columns of  $T$  are the vectors  $f_1, \dots, f_n$ ,

# Similarity transforms

$A'$  and  $A$  have the same eigenvalues

An eigenvector  $x$  of  $A$  corresponds to an eigenvector  $T^{-1}x$  of  $A'$

Rotation matrices are also a change of basis

If  $A$  is a linear transformation in  $o_0$  and  $B$  is a linear transformation in  $o_1$ , then they are related as follows:

$$B = (R_1^0)^{-1} A R_1^0$$

Ex: the frames  $o_0$  and  $o_1$  are related as follows:

$$R_1^0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

If the matrix  $A$  is also a rotation matrix  $R_{z,\theta}$  (relative to  $o_0$ ) the rotation expressed in  $o_1$  is:

$$B = (R_1^0)^{-1} A R_1^0 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix}$$

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# Compositions of rotations

w/ respect to the current frame

Ex: three frames  $o_0, o_1, o_2$

$$\left. \begin{aligned} p^0 &= R_1^0 p^1 \\ p^1 &= R_2^1 p^2 \\ p^0 &= R_2^0 p^2 \end{aligned} \right\} p^0 = R_1^0 R_2^1 p^2 \longrightarrow R_2^0 = R_1^0 R_2^1$$

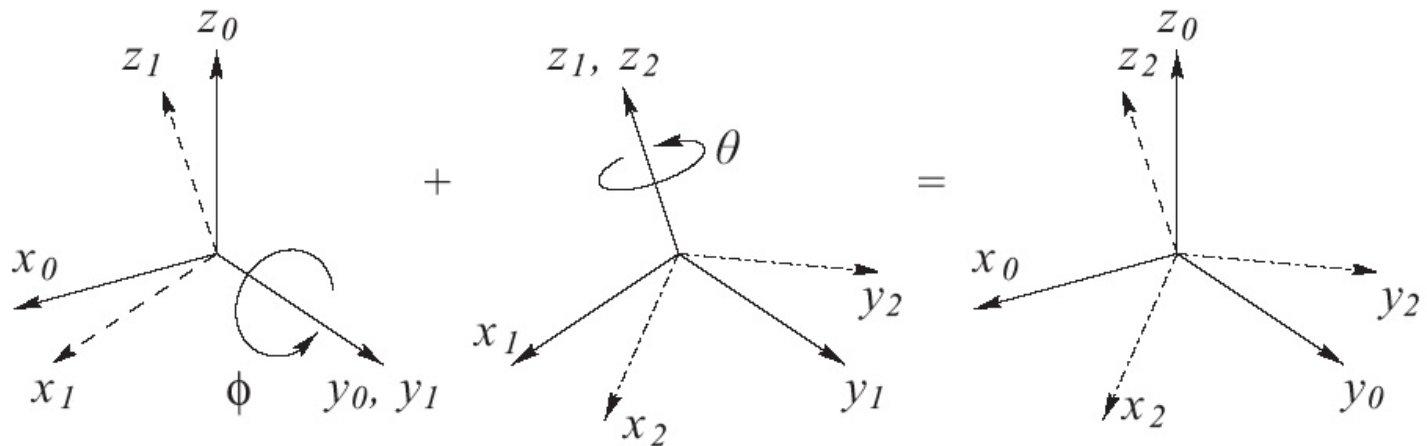
This defines the composition law for successive rotations about the **current** reference frame: post-multiplication

# Compositions of rotations

Ex:  $R$  represents rotation about the current y-axis by  $\phi$  followed by  $\theta$  about the current z-axis

$$R = R_{y,\phi} R_{z,\theta}$$

$$= \begin{bmatrix} \cos \phi & 0 & \sin \phi \\ 0 & 1 & 0 \\ -\sin \phi & 0 & \cos \phi \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos \phi \cos \theta & -\cos \phi \sin \theta & \sin \phi \\ \sin \theta & \cos \theta & 0 \\ -\sin \phi \cos \theta & \sin \phi \sin \theta & \cos \phi \end{bmatrix}$$



# Compositions of rotations

w/ respect to a fixed reference frame ( $o_0$ )

Let the rotation between two frames  $o_0$  and  $o_1$  be defined by  $R_1^0$

Let  $R$  be a desired rotation w/ respect to the fixed frame  $o_0$

Using the definition of a similarity transform, we have:

$$R_2^0 = R_1^0 \left[ (R_1^0)^{-1} R R_1^0 \right] = R R_1^0$$

This defines the composition law for successive rotations about a **fixed** reference frame: pre-multiplication

# Compositions of rotations

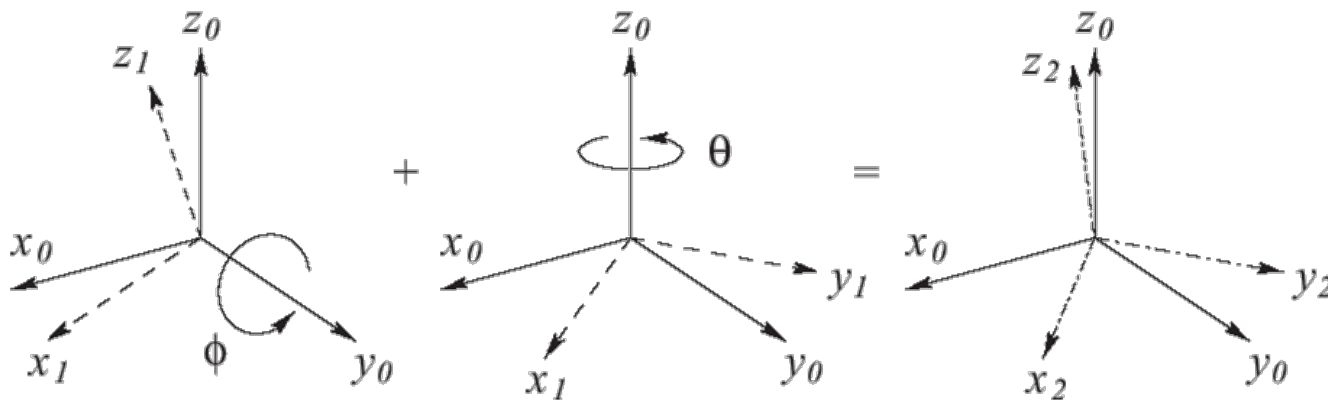
Ex: we want a rotation matrix  $R$  that is a composition of  $\phi$  about  $y_0$  ( $R_{y,\phi}$ ) and then  $\theta$  about  $z_0$  ( $R_{z,\theta}$ )

the second rotation needs to be projected back to the initial fixed frame

$$\begin{aligned} R_2^0 &= (R_{y,\theta})^{-1} R_{z,\theta} R_{y,\theta} \\ &= R_{y,-\theta} R_{z,\theta} R_{y,\theta} \end{aligned}$$

Now the combination of the two rotations is:

$$R = R_{y,\phi} [R_{y,-\phi} R_{z,\theta} R_{y,\phi}] = R_{z,\theta} R_{y,\phi}$$



# Compositions of rotations

## Summary:

Consecutive rotations w/ respect to the current reference frame:

Post-multiplying by successive rotation matrices

w/ respect to a fixed reference frame ( $o_0$ )

Pre-multiplying by successive rotation matrices

We can also have hybrid compositions of rotations with respect to the current and a fixed frame using these same rules

# Parameterizing rotations

There are three parameters that need to be specified to create arbitrary rigid body rotations

We will describe three such parameterizations:

**Euler angles**

**Roll, Pitch, Yaw angles**

**Axis/Angle**

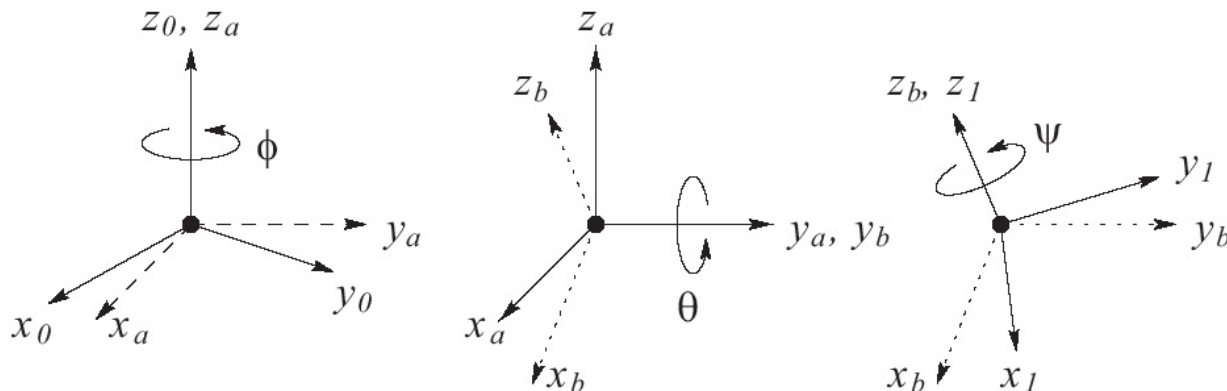
# Parameterizing rotations

## Euler angles

Rotation by  $\phi$  about the z-axis, followed by  $\theta$  about the current y-axis, then  $\psi$  about the current z-axis

$$R_{ZYZ} = R_{z,\phi} R_{y,\theta} R_{z,\psi} = \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

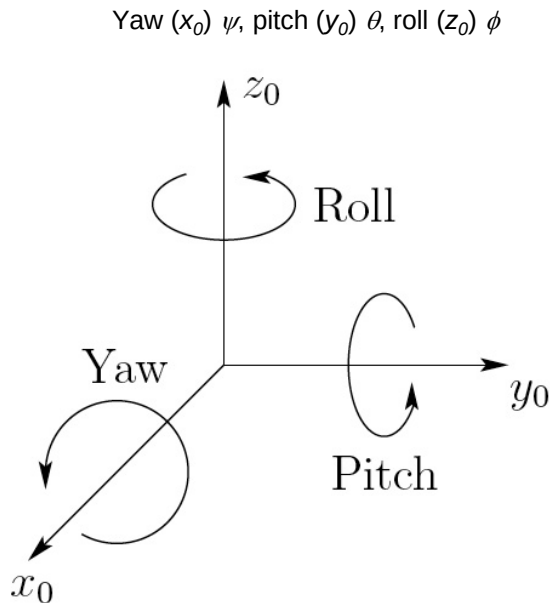
$$= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$



# Parameterizing rotations

## Roll, Pitch, Yaw angles

Three consecutive rotations about the fixed principal axes:



$$\begin{aligned}
 R_{XYZ} &= R_{z,\phi} R_{y,\theta} R_{x,\psi} \\
 &= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix} \\
 &= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}
 \end{aligned}$$

# Parameterizing rotations

## Axis/Angle representation

Any rotation matrix in  $SO(3)$  can be represented as a single rotation about a suitable axis through a set angle

For example, assume that we have a unit vector:

Given  $\theta$ , we want to derive  $R_{k,\theta}$ :

Intermediate step: project the z-axis onto  $k$ :

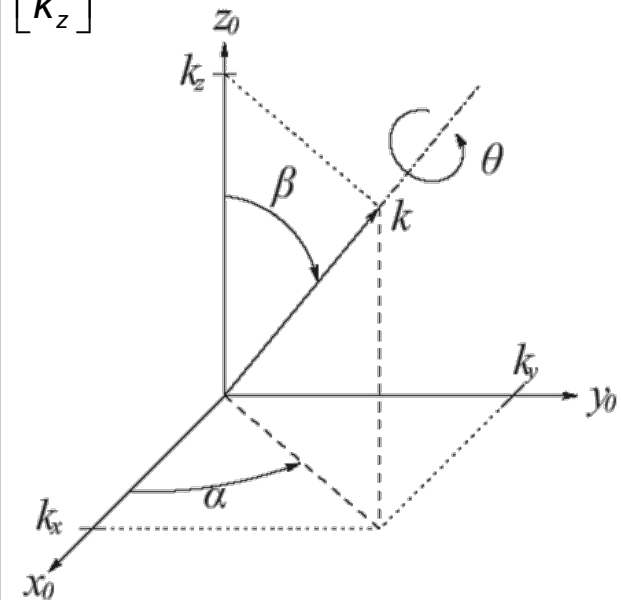
$$R_{k,\theta} = R R_{z,\theta} R^{-1}$$

Where the rotation  $R$  is given by:

$$R = R_{z,\alpha} R_{y,\beta}$$

$$\Rightarrow R_{k,\theta} = R_{z,\alpha} R_{y,\beta} R_{z,\theta} R_{y,-\beta} R_{z,-\alpha}$$

$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}$$



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# Parameterizing rotations

## Axis/Angle representation

This is given by:

$$R_{k,\theta} = \begin{bmatrix} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{bmatrix}$$

Inverse problem:

Given arbitrary  $R$ , find  $k$  and  $\theta$

$$\theta = \cos^{-1} \left( \frac{\text{Tr}(R) - 1}{2} \right)$$

$$\hat{k} = \frac{1}{2 \sin \theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

# Alternative approach

Rotation matrices in  $SO(3)$  can be derived using an alternate method using the matrix exponential

Assume that you have a unit vector,  $\omega$ , that represents an axis of rotation

Take a vector  $q(t)$  and rotate it about  $\omega$  with unity velocity

This gives:

$$\begin{aligned}\dot{\vec{q}}(t) &= \omega \times \vec{q}(t) \\ &= \hat{\omega} \vec{q}(t)\end{aligned}$$

Where the 'cross-product' matrix is given as follows:

Now integrate to find the vector  $q(t)$ :

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Rotating for  $\theta$  units of time gives:

$$\begin{aligned}\vec{q}(t) &= e^{\hat{\omega}t} \vec{q}(0) \\ \vec{q}(t) &= e^{\hat{\omega}\theta} \vec{q}(0) = R_{\omega,\theta} \vec{q}(0)\end{aligned}$$

Thus the rotation about  $\omega$  can be represented by the matrix exponential

For notation, we can call:

$$\hat{\omega} \in so(3)$$

# Rigid motions

Rigid motion is a combination of rotation and translation

Defined by a rotation matrix ( $R$ ) and a displacement vector ( $d$ )

$$R \in SO(3)$$

$$d \in \mathbf{R}^3$$

the group of all rigid motions ( $d, R$ ) is known as the **Special Euclidean group**,  $SE(3)$

$$SE(3) = \mathbf{R}^3 \times SO(3)$$

Consider three frames,  $o_0$ ,  $o_1$ , and  $o_2$  and corresponding rotation matrices  $R_2^1$ , and  $R_1^0$

Let  $d_2^1$  be the vector from the origin  $o_1$  to  $o_2$ ,  $d_1^0$  from  $o_0$  to  $o_1$

For a point  $p^2$  attached to  $o_2$ , we can represent this vector in frames  $o_0$  and  $o_1$ :

$$p^1 = R_2^1 p^2 + d_2^1$$

$$p^0 = R_1^0 p^1 + d_1^0$$

$$= R_1^0 (R_2^1 p^2 + d_2^1) + d_1^0$$

$$= R_1^0 R_2^1 p^2 + R_1^0 d_2^1 + d_1^0$$

# Homogeneous transforms

We can represent rigid motions (rotations and translations) as matrix multiplication

Define:

$$H_1^0 = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix}$$

$$H_2^1 = \begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix}$$

Now the point  $p_2$  can be represented in frame  $o_0$ :

$$P^0 = H_1^0 H_2^1 P^2$$

Where the  $P^0$  and  $P^2$  are:

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix}, P^2 = \begin{bmatrix} p^2 \\ 1 \end{bmatrix}$$

# Homogeneous transforms

The matrix multiplication  $H$  is known as a **homogeneous transform** and we note that

$$H \in SE(3)$$

Inverse transforms:

$$H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}$$

# Homogeneous transforms

## Basic transforms:

Three pure translation, three pure rotation

$$\mathbf{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{Rot}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{Rot}_{y,\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{Rot}_{z,\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$