



UiO : **Department of Technology Systems**
University of Oslo

Lecture 11 - Control Theory

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Control of Manipulators and Mobile Robots

UNIK4490/TEK4030



Lecture overview

- General introduction to control theory
 - Motivation
 - Self regulating systems (pendulum at equilibrium)
 - Unstable systems (car speed)
 - Open and closed loop systems
 - Open loop (washing machine)
 - Feed forward (car speed – incline as disturbance model)
 - Closed loop
 - Feedback (cruise control)
 - Stability
 - [Stable systems](#)
 - [Unstable systems](#)

Definition input/output stability

- Asymptotically stable if:
 - $y \rightarrow 0$ when $t \rightarrow \infty$ and u has a finite duration and amplitude
- Marginally stable if:
 - $|y| < \infty$ for all $t \geq 0$ and u has a finite duration and amplitude
- Unstable otherwise

Robot Control

- We want to control the joint positions (configuration) of a robot
 - Therefore we need to figure out how we can determine the motor/actuator inputs so as to command the robot to a desired configuration
- In general, the input (voltage/current) does not create instantaneous motion to a desired configuration because of the robots dynamical properties
- There are also other real world elements that affect the robots motion like backlash (clearance between gear teeths - causes hysteresis) and the properties of the motor/actuator

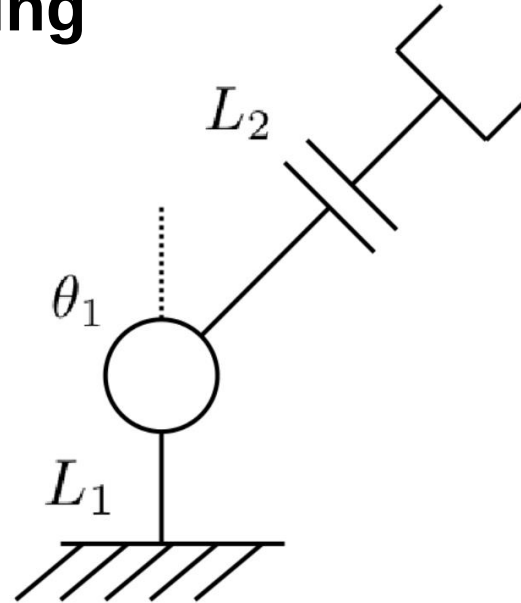
Robot Control

- We shall focus on single input single output systems (SISO) meaning that we only look at each joint by itself
- We therefore assume that each joint is affected only by itself
- This means that the contributions from the other joints are treated as a disturbance
- Before we can start to control a joint, we need to model the joint and look at its properties.

Robot control - Modelling

- From dynamics we have the ordinary differential equation (ODE) for the robot. We have n non-linear equations as a result of n joint variables, all expressed in the time domain.
- These n equations often depends on each other
- We want to transform these equations into n linear, independent equations
- We do this by treating the non-linear effects and the dependence of other joints as disturbance

Example – Robot modeling



The dynamic equations are given as

$$mL_2^2\ddot{\theta}_1 + 2mL_2\dot{L}_2\dot{\theta}_1 - mgs_1L_2 = \tau_1$$
$$m\ddot{L}_2 - mL_2\dot{\theta}_1^2 - mgc_1 = F_2$$

We shall now give a linearization of the first equation

Robot control - Modeling

- We now have a set of independent linear equations
- We want to analyse the properties of these systems
- Laplace transform can be used to transform the equations into the frequency domain
 - This transforms the equations from a ordinary differential equations into a linear equations, which easier to solve
 - We can analyse important system properties in the frequency domain, such as stability and step response
- Block diagrams can be used to visualize the equations for the system and possible feedback loops

Laplace transform

- The Laplace transform is defined as follows:

$$x(s) = \int_0^{\infty} e^{-st} x(t) dt$$

- For example, Laplace transform of a derivative:

$$L\{\dot{x}(t)\} = L\left\{\frac{dx(t)}{dt}\right\} = \int_0^{\infty} e^{-st} \frac{dx(t)}{dt} dt$$

- Integrating by parts:

$$\begin{aligned} L\left\{\frac{dx(t)}{dt}\right\} &= e^{-st} x(t) \Big|_0^{\infty} + s \int_0^{\infty} e^{-st} x(t) dt \\ &= sx(s) - x(0) \end{aligned}$$

Laplace transform

- Similarly, Laplace transform of a second derivative:

$$L\{\ddot{x}(t)\} = L\left\{\frac{d^2x(t)}{dt^2}\right\} = \int_0^{\infty} e^{-st} \frac{d^2x(t)}{dt^2} dt = s^2x(s) - sx(0) - \dot{x}(0)$$

- Thus, if we have a generic 2nd order system described by the following ODE:

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = F(t)$$

- And we want to get a transfer function representation of the system, take the Laplace transform of both sides:

$$\begin{aligned} mL\{\ddot{x}(t)\} + bL\{\dot{x}(t)\} + kL\{x(t)\} &= L\{F(t)\} \\ m(s^2x(s) - sx(0) - \dot{x}(0)) + b(sx(s) - x(0)) + kx(s) &= F(s) \end{aligned}$$

Laplace transform

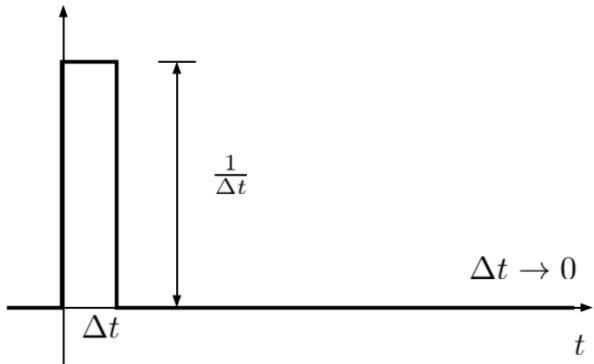
- Continuing:

$$(ms^2 + bs + k)x(s) = F(s) + m\dot{x}(0) + (ms + c)x(0)$$

- The *transient response* is the solution of the above ODE if the *forcing function* $F(t) = 0$
- The steady state response is the solution of the above equation if the initial conditions are zero
- This yields the equation

$$(ms^2 + bs + k)x(s) = F(s)$$

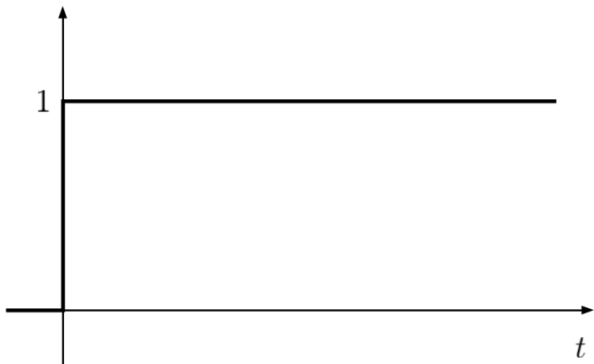
Common Laplace functions



$$\delta(t) = \lim_{\Delta t \rightarrow 0} g(t, \Delta t)$$

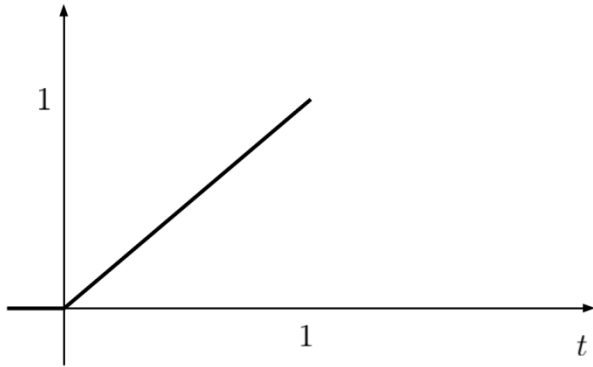
$$g(t, \Delta t) = \begin{cases} \frac{1}{\Delta t} & , 0 < t < \Delta t \\ 0 & , t < 0, t > \Delta t \end{cases}$$

$$1$$



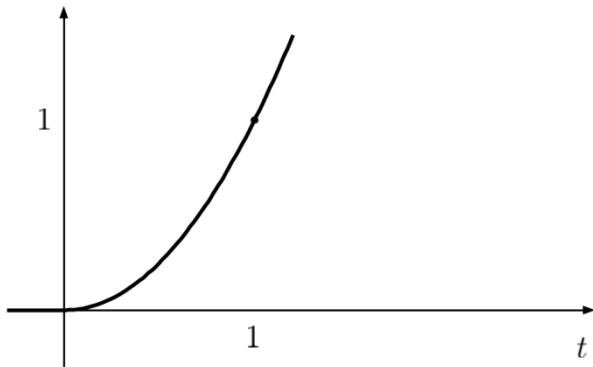
$$f(t) = \begin{cases} 1 & , t \geq 0 \\ 0 & , t < 0 \end{cases}$$

$$\frac{1}{s}$$



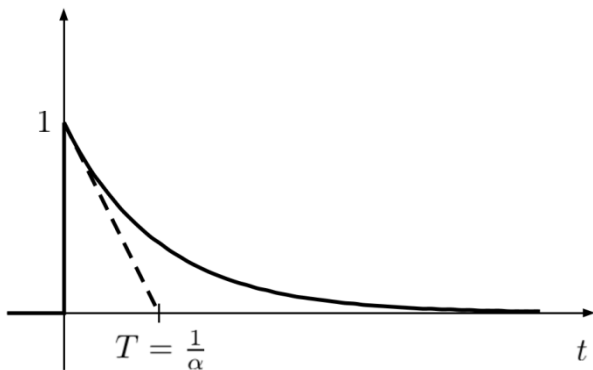
$$f(t) = \begin{cases} t & , t \geq 0 \\ 0 & , t < 0 \end{cases}$$

$$\frac{1}{s^2}$$



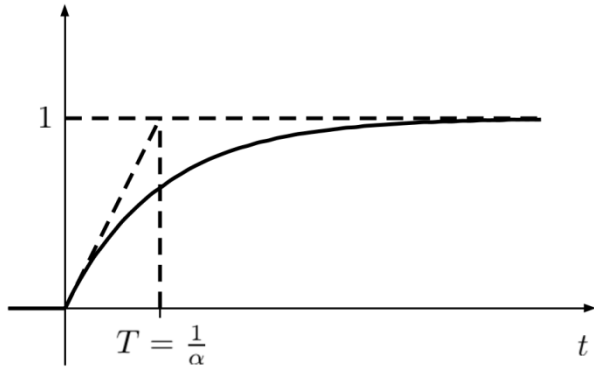
$$f(t) = \begin{cases} t^2 & , t \geq 0 \\ 0 & , t < 0 \end{cases}$$

$$\frac{1}{s^3}$$



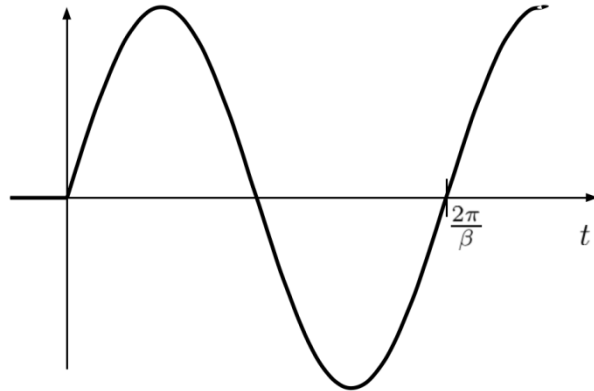
$$f(t) = \begin{cases} e^{-\alpha t} & , t \geq 0 \\ 0 & , t < 0 \end{cases}$$

$$\frac{1}{s + \alpha} = \frac{T}{1 + Ts}$$



$$f(t) = \begin{cases} 1 - e^{-\alpha t} & , t \geq 0 \\ 0 & , t < 0 \end{cases}$$

$$\frac{\alpha}{s(s + \alpha)} = \frac{1}{s(1 + Ts)}$$



$$f(t) = \begin{cases} \sin \beta t & , t \geq 0 \\ 0 & , t < 0 \end{cases}$$

$$\frac{\beta}{s^2 + \beta^2} = \frac{\beta}{(s + j\beta)(s - j\beta)}$$

Review of the Laplace transform

- Properties of the Laplace transform
 - Takes an ODE to a algebraic equation
 - Differentiation in the time domain is multiplication by s in the Laplace domain
 - Integration in the time domain is multiplication by $1/s$ in the Laplace domain
 - Considers initial conditions
 - i.e. transient and steady-state response
 - The Laplace transform is a linear operator

Transfer functions

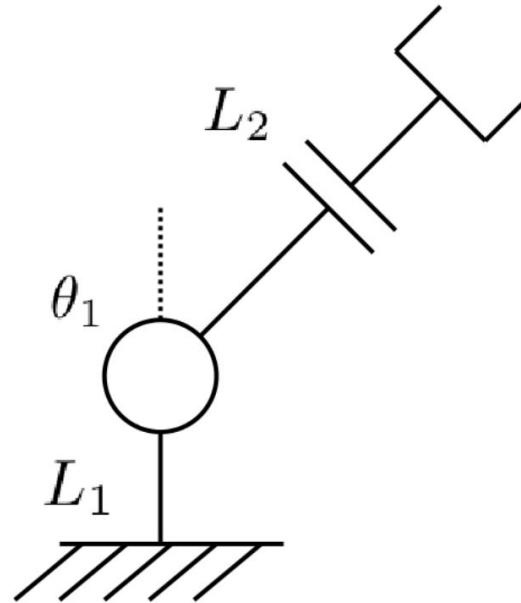
- When all initial conditions of a Laplace transform are zero, the response $Y(s)$ of a linear system is given by its input $X(s)$ and its transfer function $H(s)$

$$Y(s) = H(s) X(s)$$

$$\frac{Y(s)}{X(s)} = H(s)$$

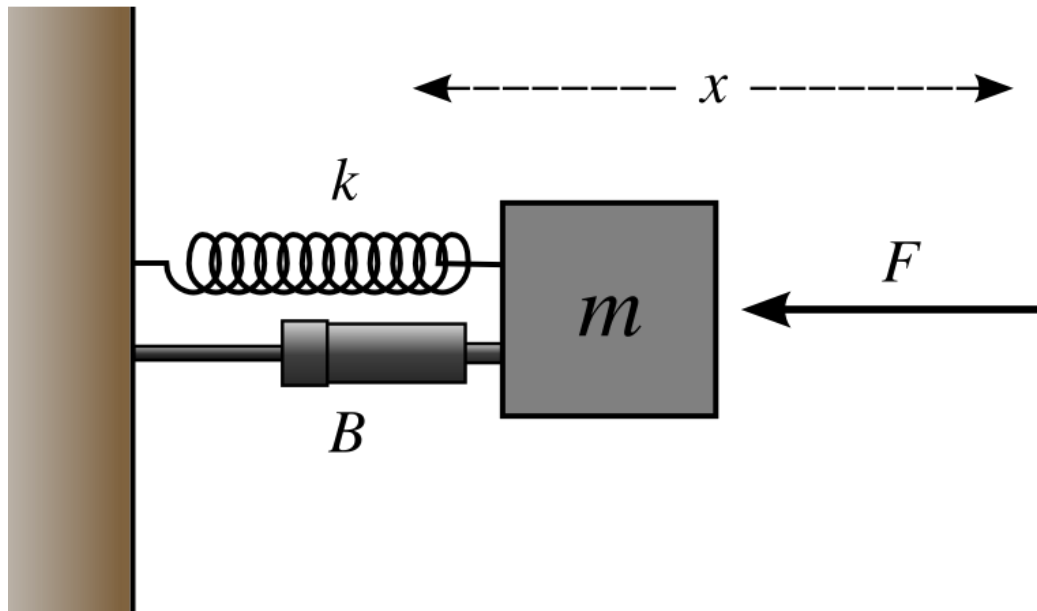
- The denominator ($X(s)$) of the transfer function is called the characteristic polynomial

Example cont. – Robot modeling



We can now derive the transfer functions for the robot in the frequency domain

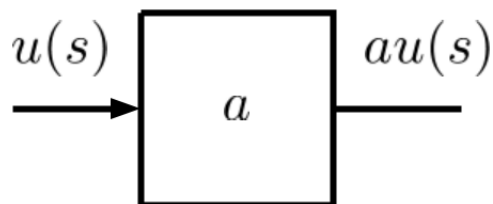
Example – Mass spring damper system



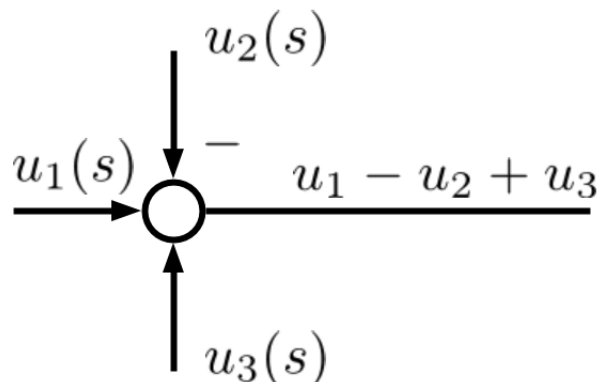
Block diagrams

- A block diagram visualizes one or more equations
- Makes it easier to see feedback and feed forward loops

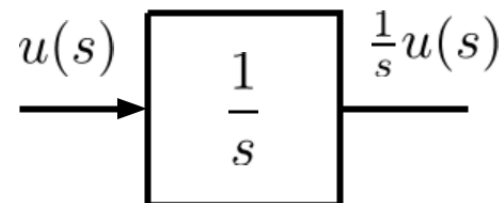
Block diagram symbols



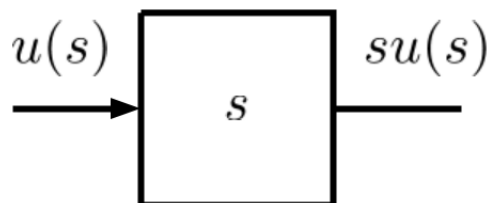
Multiplication
with constant



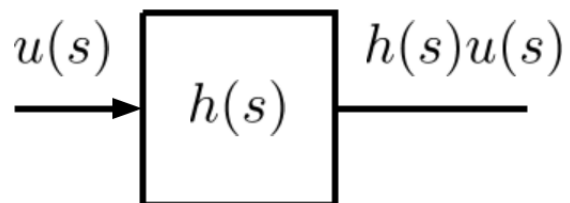
Addition/subtraction



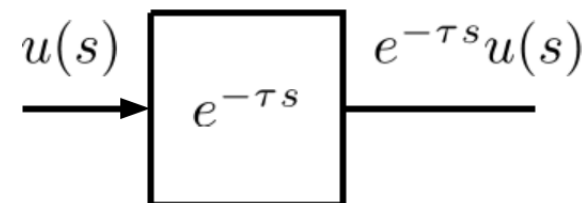
Integral



Derivative

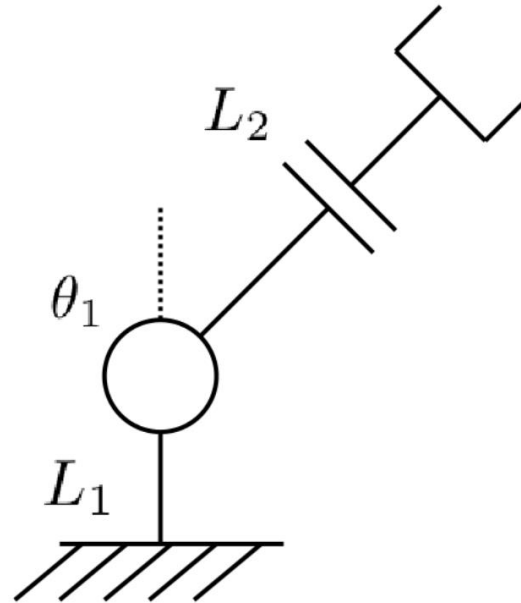


Transfer function



Time delay

Example cont. – Robot modeling



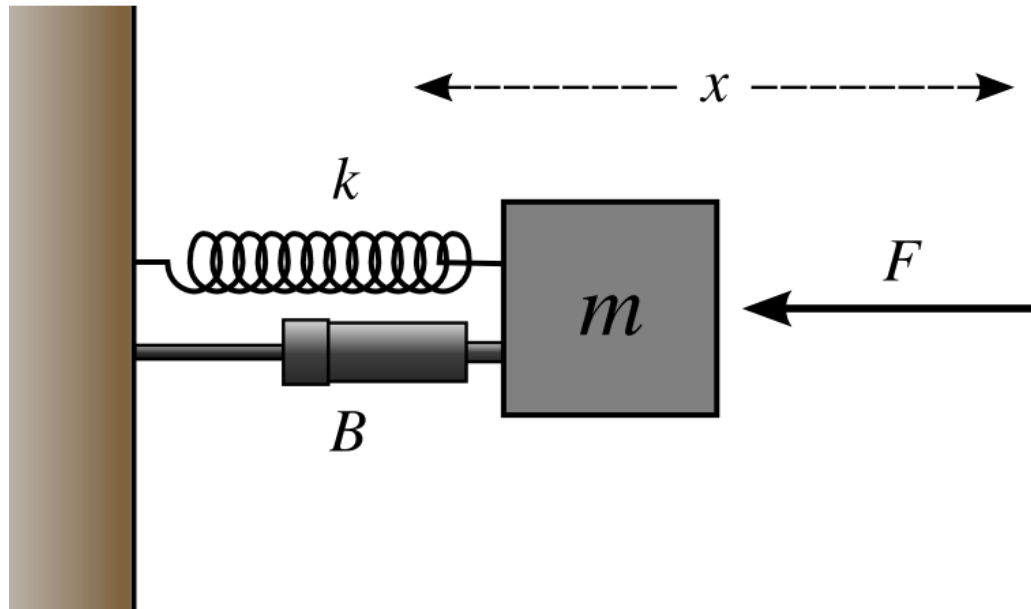
Drawing a block diagram

The transfer function as given as:

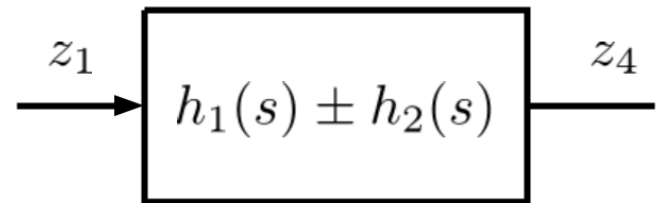
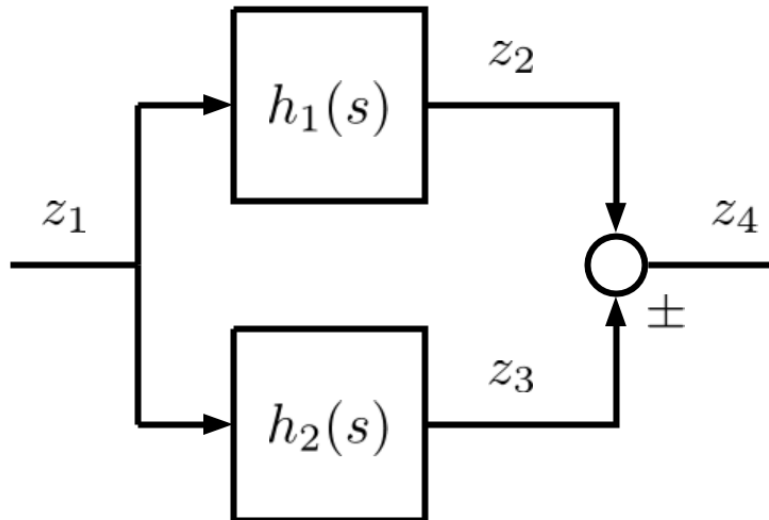
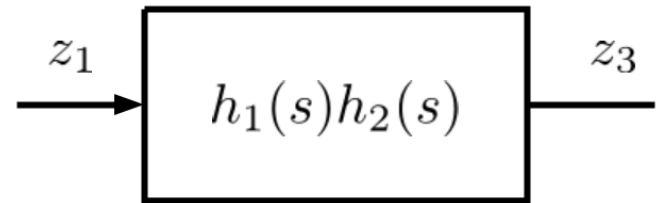
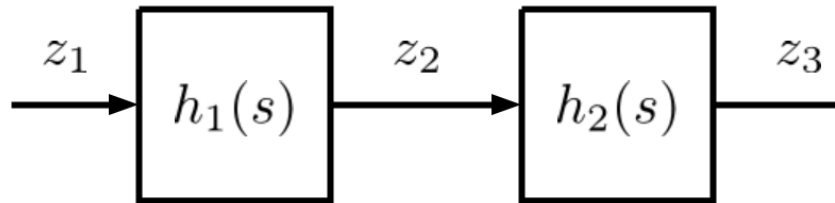
$$Js^2\theta_1 - G\theta_1 + D = \tau_1$$

Drawing a block diagram of a mass spring damper system

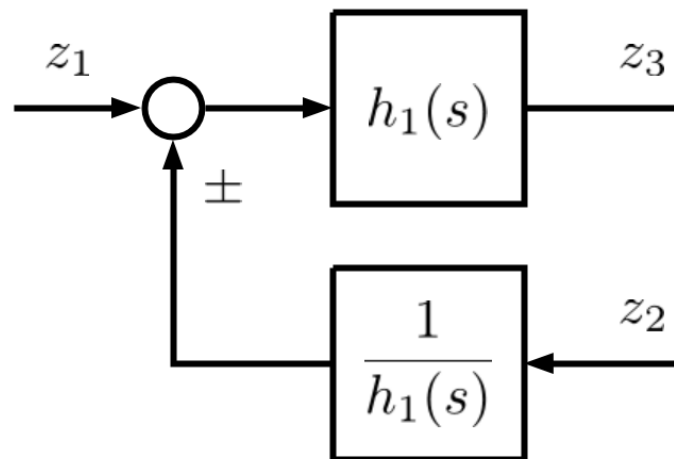
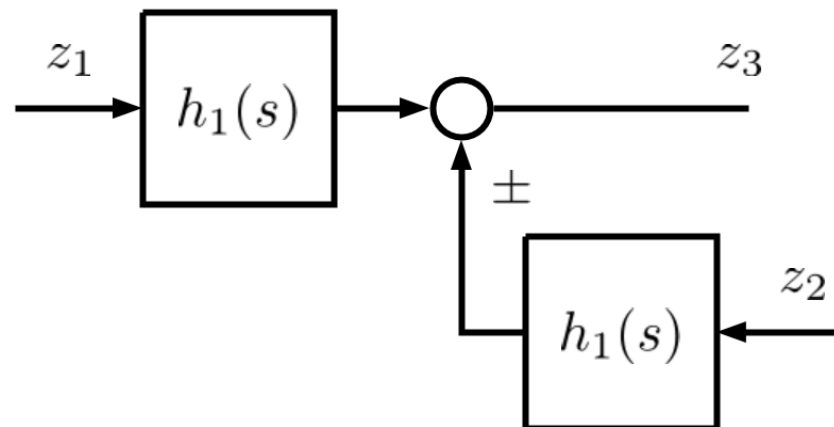
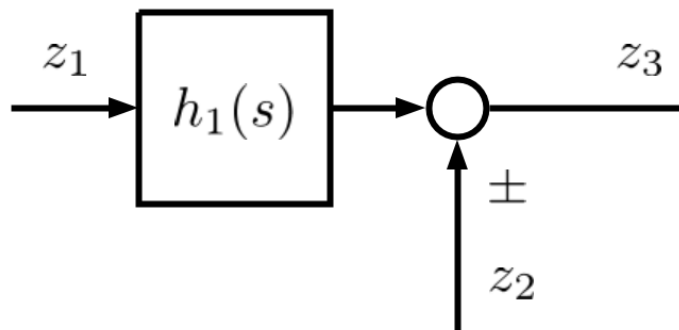
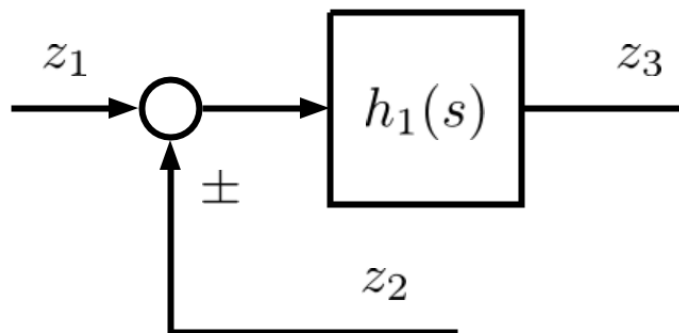
- $x s^2 = \frac{1}{m} (u - f x s - k x)$



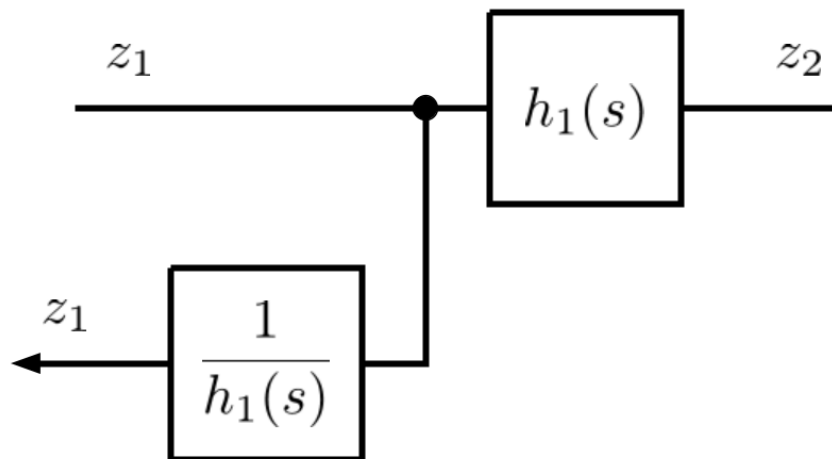
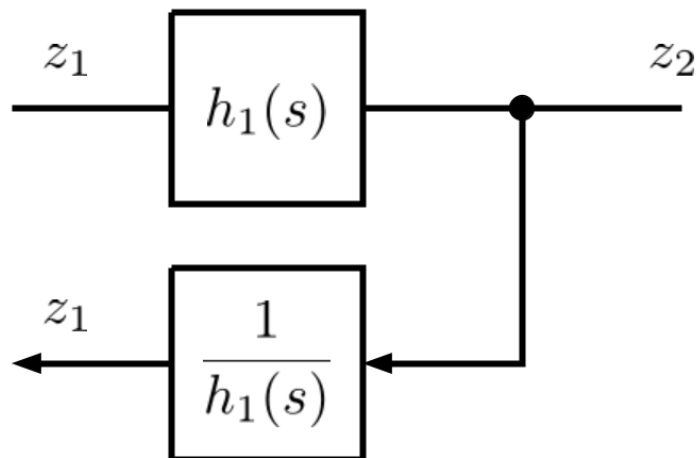
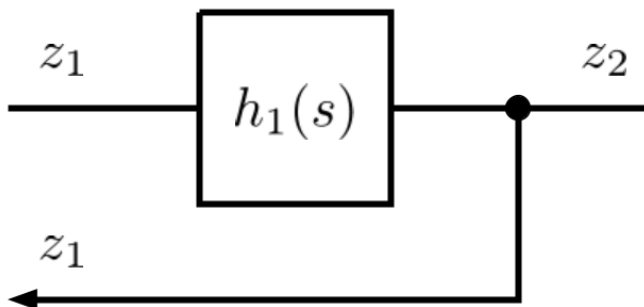
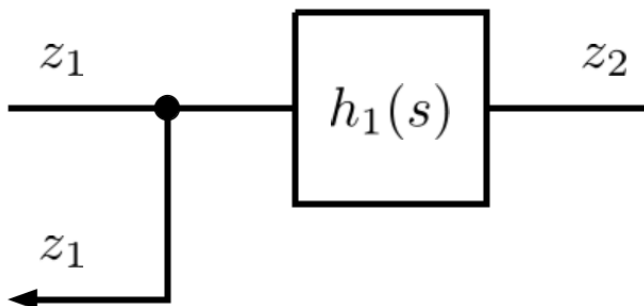
Manipulation of block diagrams



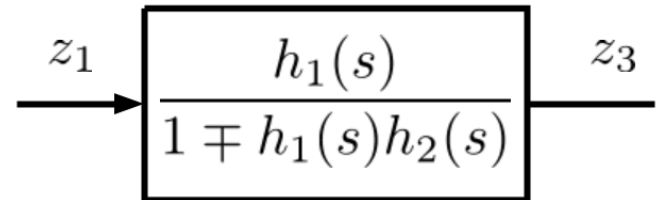
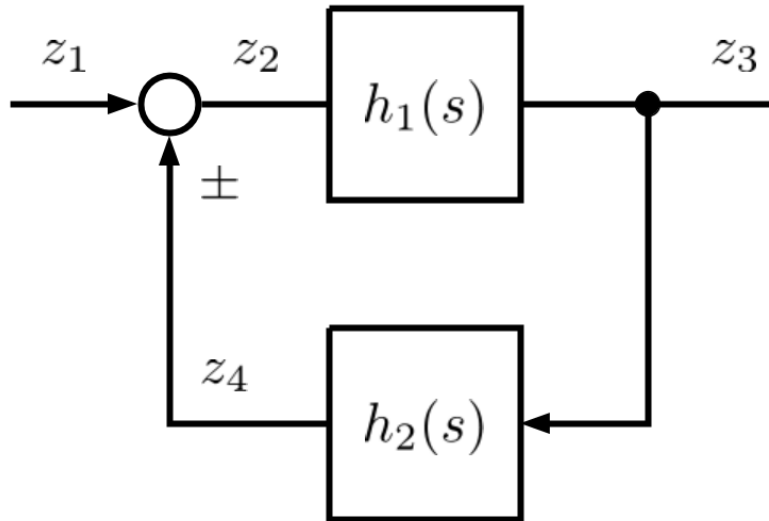
Manipulation of block diagrams



Manipulation of block diagrams



Manipulation of block diagrams



Prove on blackboard

Zeros and poles of the transfer functions

- For rational transferfunctions we denote the roots of the nominator zeros and roots of the denominator poles
- The poles gives important characteristics about the transfer function

$$h(s) = \frac{\rho_p s^p + \dots + \rho_1 s^1 + \rho_0}{s^n + \alpha_{n-1} s^{n-1} + \dots + \alpha_1 s + \alpha_0}$$
$$h(s) = \frac{\rho_p (s - v_1) \dots (s - v_p)}{(s - \lambda_1) \dots (s - \lambda_n)}$$

- We call the polynomial in the denominator the characteristic polynomial

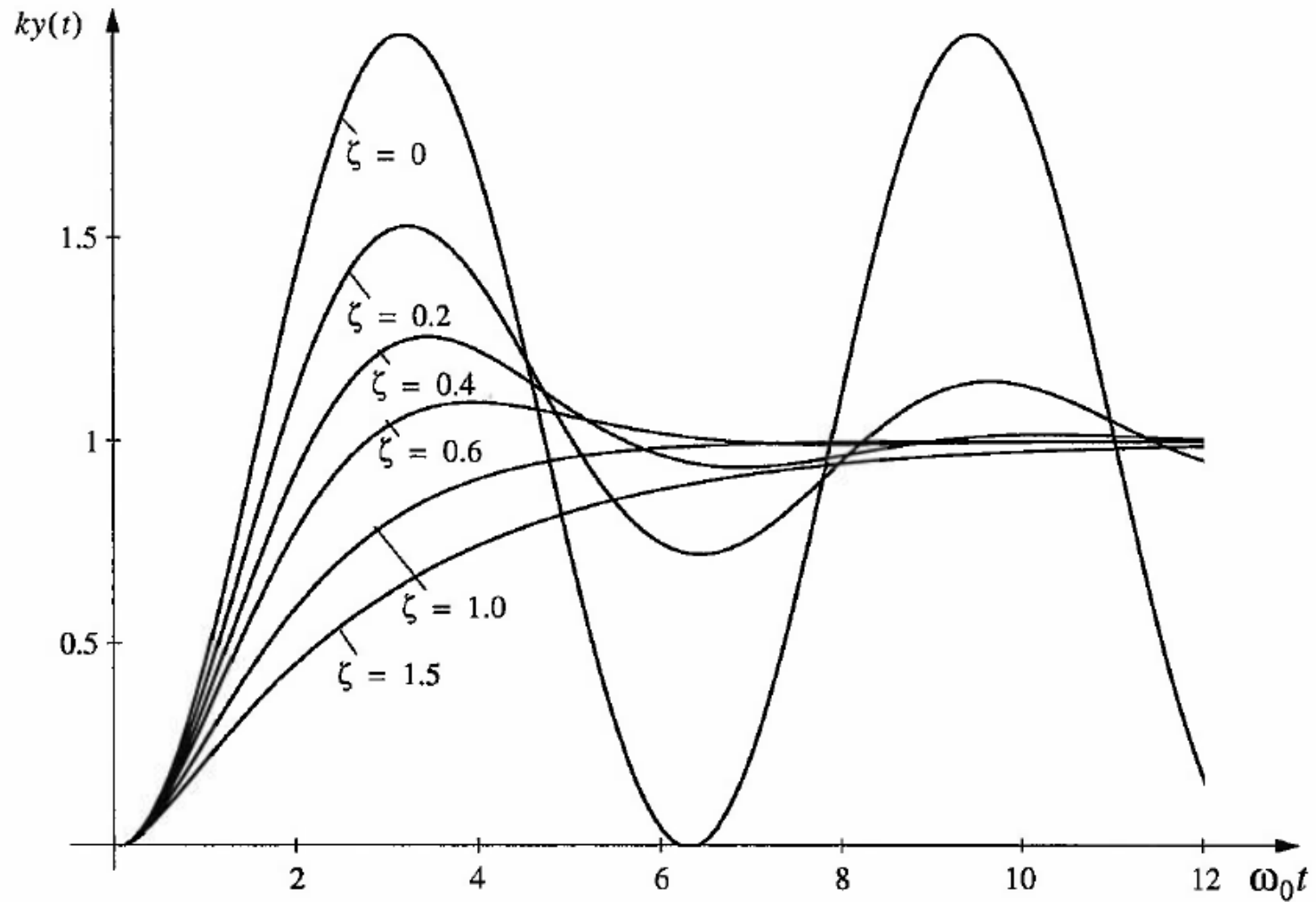
Examples – Finding roots and zeros

- Example 1: Given transfer function
- Example 2: Mass-spring-damper system

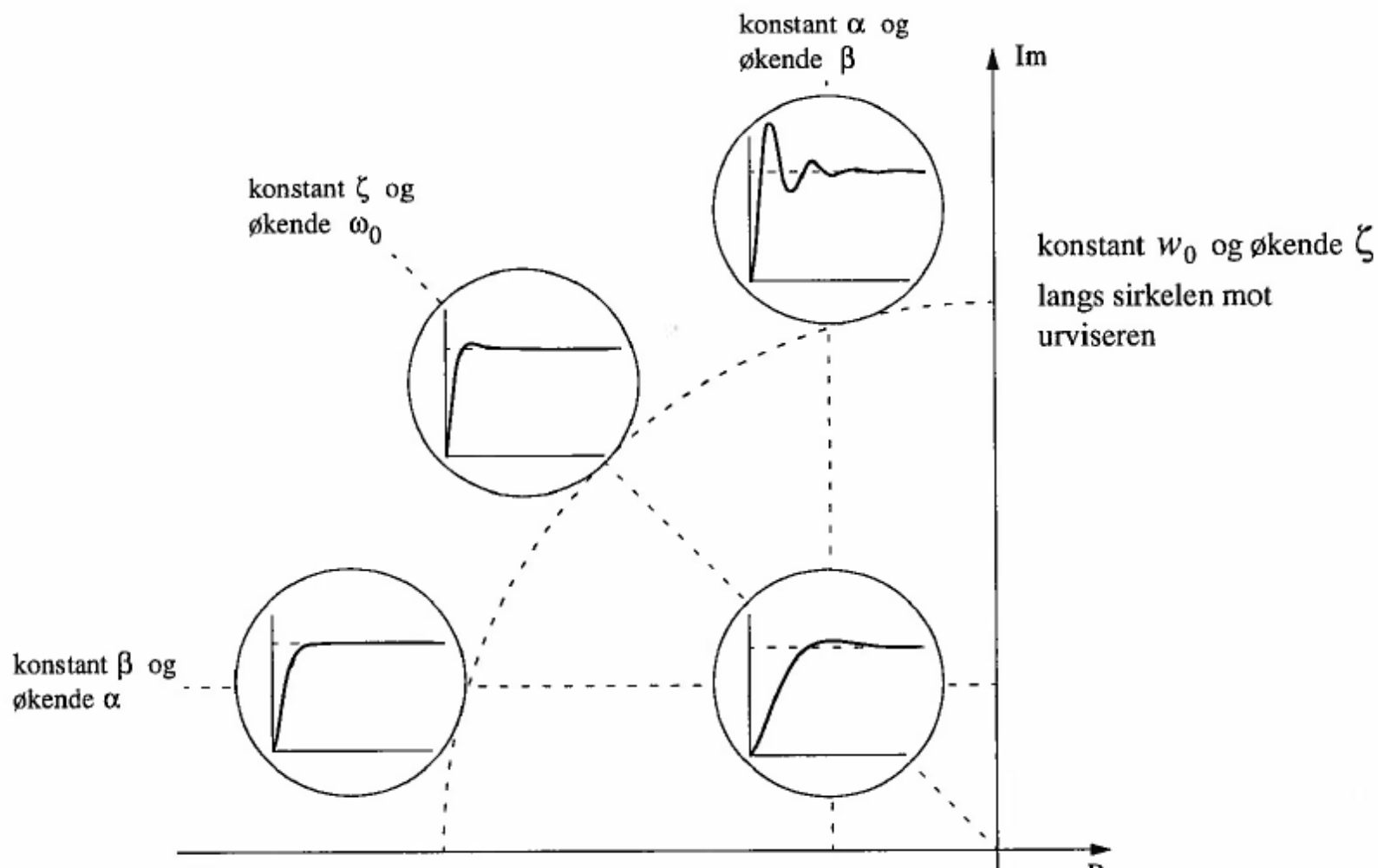
Example cases

- Three cases depending on the poles
- Case I: Poles are real and distinct
 - Over-damped system
- Case II: Poles are real and equal
 - Critically damped system
- Case III: Poles are complex conjugates
 - Under-damped system

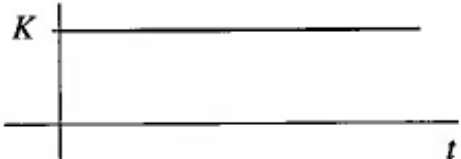
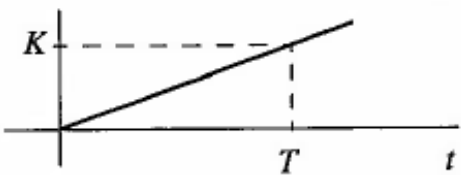
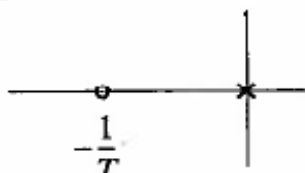
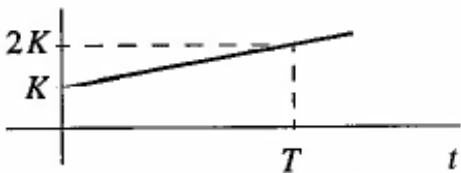
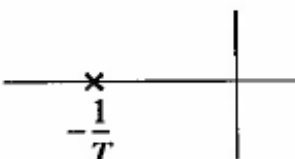
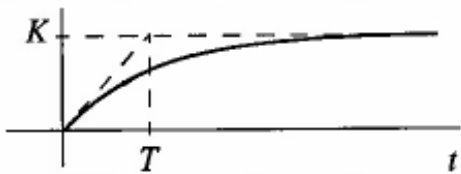
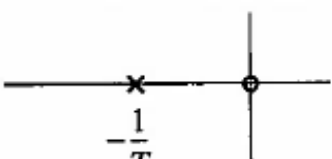
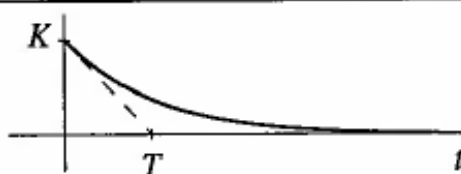
Time responses

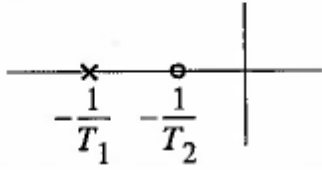
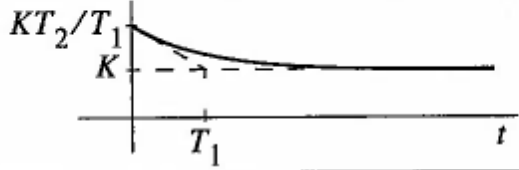
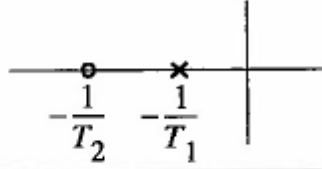
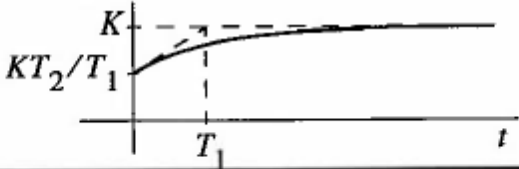
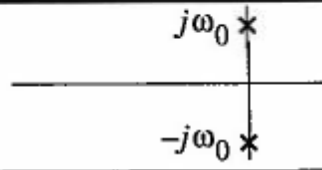
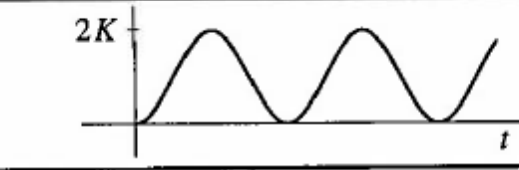
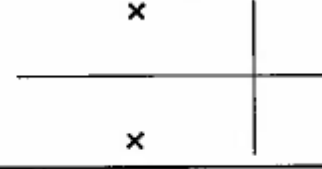
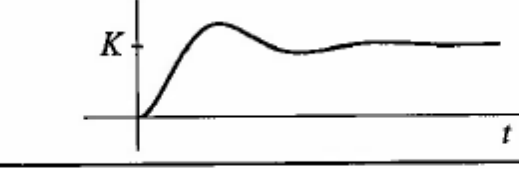
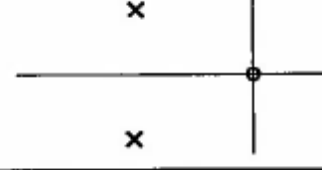
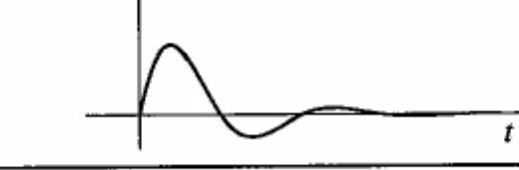
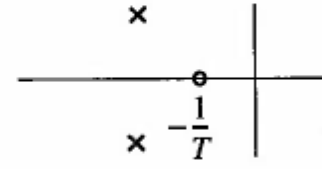
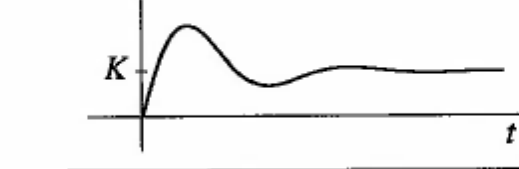


Effect of changes in poles



Common transfer functions and their poles and step responses

Transfer-funksjon $h(s)$	Nullpunkter og poler	Sprangrespons
K		
$K \frac{1}{Ts}$		
$K \frac{1+Ts}{Ts}$		
$K \frac{1}{1+Ts}$		
$K \frac{Ts}{1+Ts}$		

Transfer-funksjon $h(s)$	Nullpunkter og poler	Sprangrepons
$K \frac{1+T_2 s}{1+T_1 s}$ $T_2 > T_1$		
$K \frac{1+T_2 s}{1+T_1 s}$ $T_2 < T_1$		
$\frac{K}{1 + \left(\frac{s}{\omega_0}\right)^2}$		
$\frac{K}{1 + 2\zeta \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$		
$\frac{Ks}{1 + 2\zeta \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$		
$\frac{K(1+Ts)}{1 + 2\zeta \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$		

Root locus plots

- Instead of using constant values in the transfer function we now assume that we can *vary* one (or more) parameters
- Changing this parameter will move the poles and zeros *in the complex plane*
- The paths of zeros and poles in the complex plane as a function of changed controller parameters are called **root locus plots**
- We can investigate the stability of our system by looking at how these poles moves based on the control parameters

Example - Root Locus Plot

Drawing on blackboard

Stability – Frequency domain

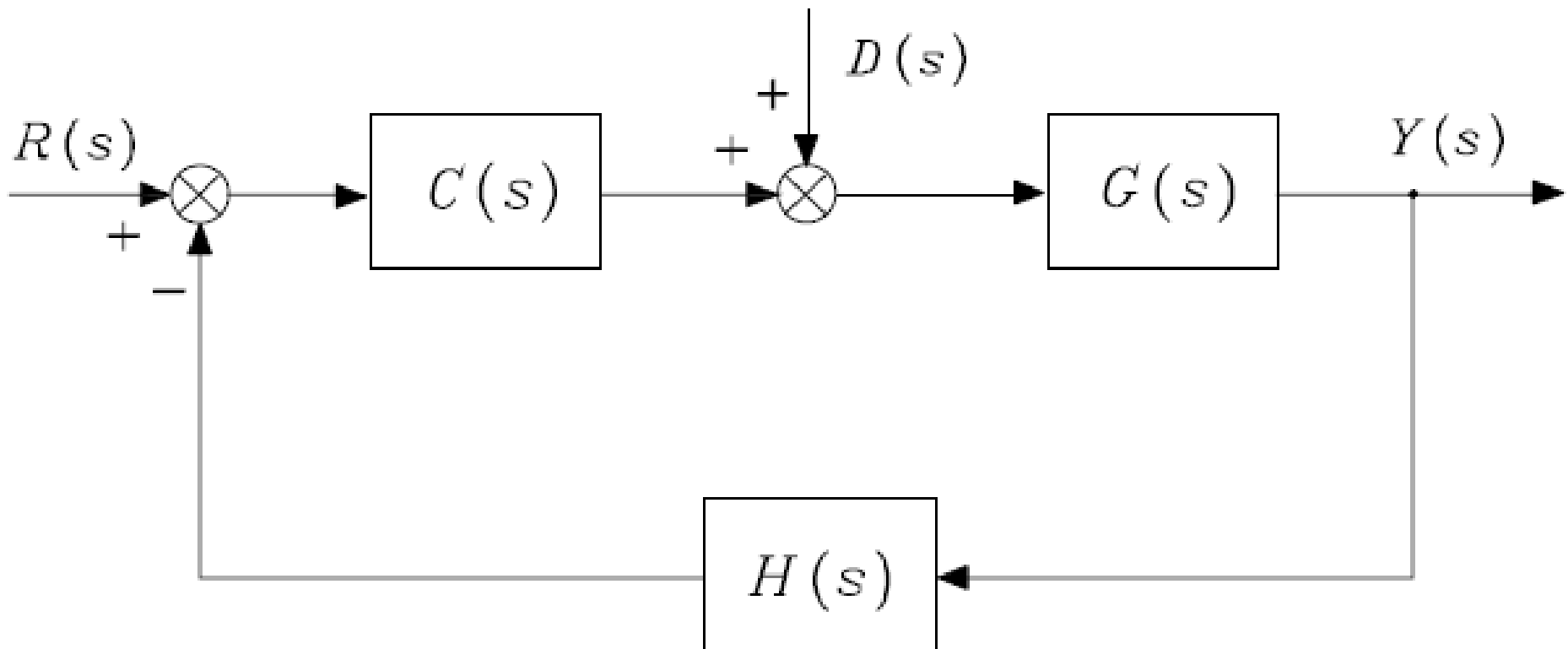
- Find the poles (λ_i) of the transfer function
- If $Re(\lambda_i) < 0$ for all λ_i in $H(s)$ the system is *asymptotically stable*
- If one or more poles has $Re(\lambda_i) = 0$, but they are not in the same point the system is *marginally stable*
- If one or more poles has $Re(\lambda_i) > 0$ the system is *unstable*

Example - Stability

Drawing on blackboard

Feedback systems

- $G(s)$ – System Model
- $C(s)$ – Controller
- $D(s)$ – Disturbance
- $H(s)$ – Transducer (sensor model)
- $R(s)$ – Reference Input
- $Y(s)$ – Output Variable



Feedback systems

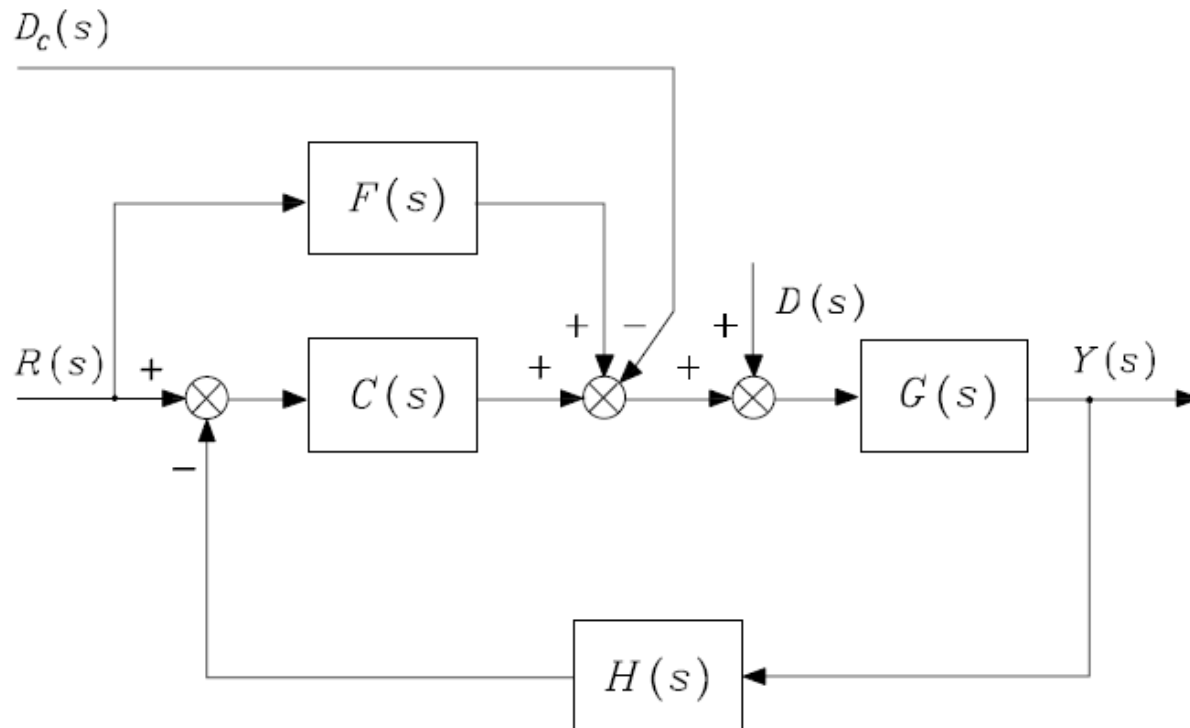
$$Y(s) = W(s)R(s) + W_D(s)D(s),$$

$$W(s) = \frac{C(s)G(s)}{1 + C(s)G(s)H(s)}$$

$$W_D(s) = \frac{G(s)}{1 + C(s)G(s)H(s)}$$

Feedback + Feedforward

- We want to regulate the system modeling error, therefore we add the feed forward parts $F(s)$ and $D_c(s)$, where $D_c(s)$ is a model of the systems disturbances.
- Assuming we can model the system accurately a convenient choice is $F(s) = 1/G(s)$
- This decreases the systems time response (feedback systems can be slower)



Feedback + Feedforward

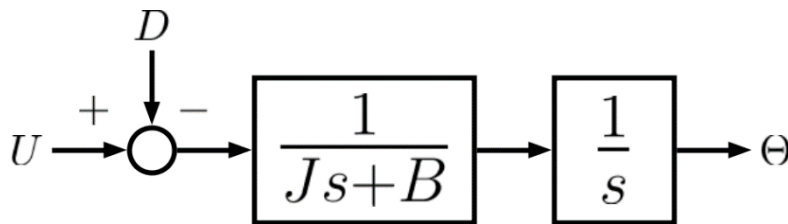
- The systems transfer function is then given as

$$Y(s) = \left(\frac{C(s)G(s)}{1 + C(s)G(s)H(s)} + \frac{F(s)G(s)}{1 + C(s)G(s)H(s)} \right) R(s) \quad (\text{C.8})$$
$$+ \frac{G(s)}{1 + C(s)G(s)H(s)} (D(s) - D_c(s)).$$

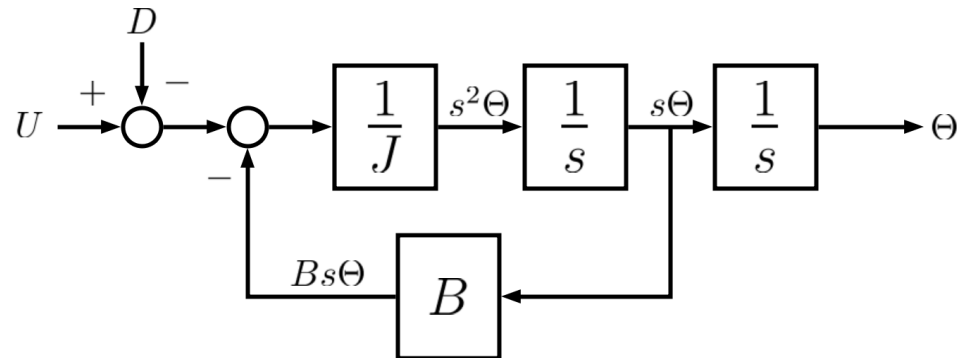
Setpoint Controllers

- We will discuss three common controllers: P, PD and PID
 - All controllers attempt to drive the error (between a desired trajectory and the actual trajectory) to zero
- The system ($G(s)$) can have any dynamics, but we will use the following system as an example

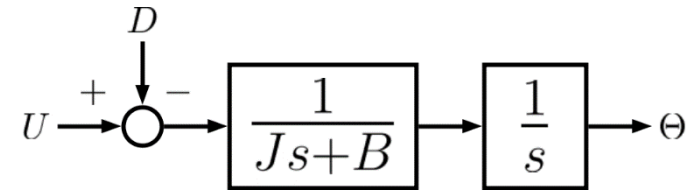
$$\theta = \frac{U - D}{(Js^2 + Bs)}$$



Compact block diagram



Block diagram with basic building blocks

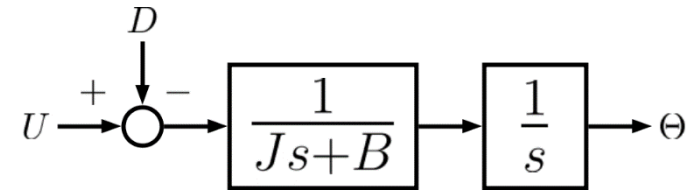


Setpoint Controllers – System Model

- A generic robot model is given as

$$J(q)\ddot{q} + C(q, \dot{q})\dot{q} + B\dot{q} + g(q) = \tau$$

- $J(q)\ddot{q}$ - Inertial forces
- $C(q, \dot{q})\dot{q}$ - Coriolis and centrifugal forces
- $B\dot{q}$ - Viscous friction (damping)
- $g(q)$ - Gravitational forces
- τ - Torque/Force from actuators



Setpoint Controllers – System Model

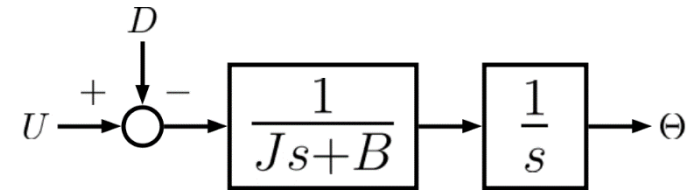
- In our example we assume that the following are treated as a disturbance D
 - Coriolis and centrifugal forces
 - Gravitational forces
 - Coupling between joints ($J(q)\ddot{q} \rightarrow J\ddot{q}$, Inertia is no longer dependent on the joint variables)

$$J(q)\ddot{q} + C(q, \dot{q})\dot{q} + B\dot{q} + g(q) = \tau$$

$$J\ddot{q} + B\dot{q} + D = \tau$$

- Transforming into the frequency domain (with Laplace) gives (remember that θ is our joint variable q)

$$Js^2\theta + Bs\theta + D = \tau$$



Setpoint Controllers – Motivation

- We will now look at different controllers for this system.
- We want the error between the reference (desired) value and the actual output value to go to zero
- The error is defined as $e(t) = \theta^d(t) - \theta(t)$
- The controller use the error $e(t)$ to calculate its output, also called *control effort*
- We denote the *control effort* as u (“U” in the block diagram above)
- We will look at the following controllers:
 1. Proportional (P) controller
 2. Proportional Derivative (PD) controller
 3. Proportional Integral Derivative (PID) controller

Proportional (P) Controller

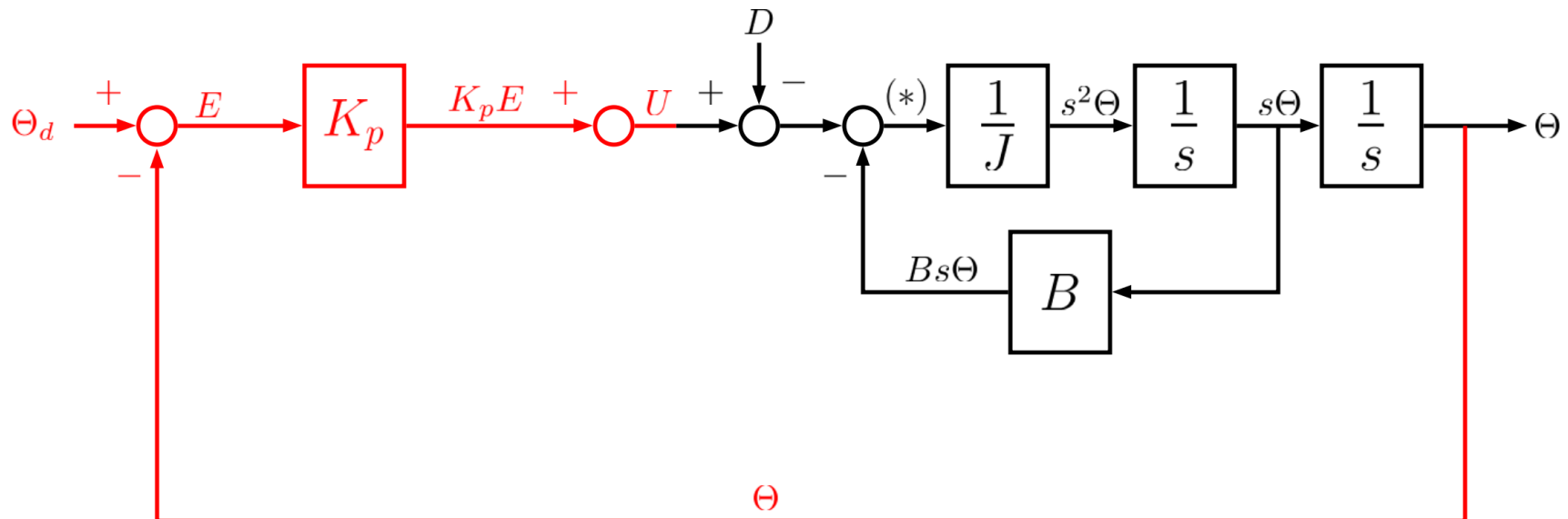
- Control law: $U(t) = K_p e(t)$

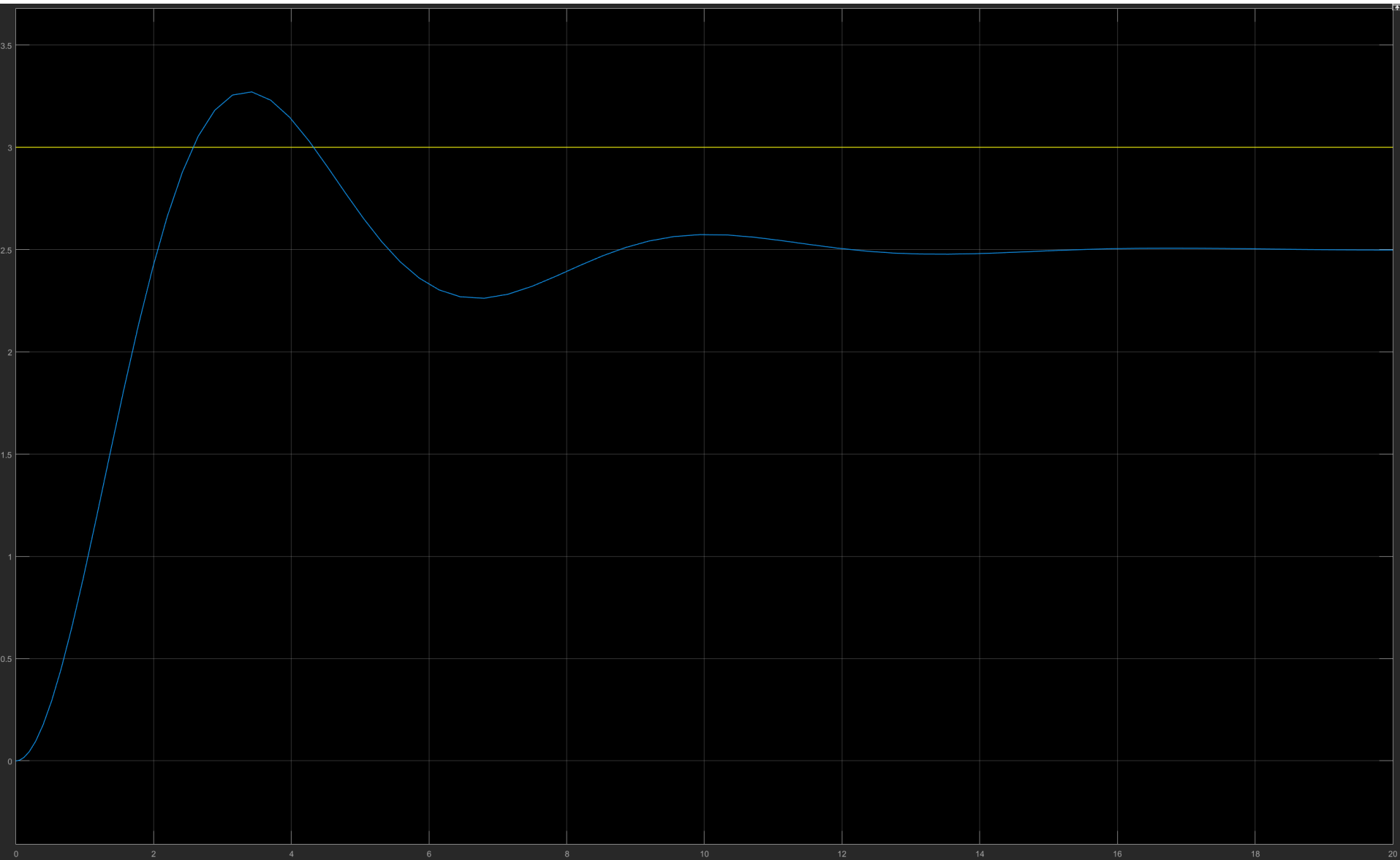
• Where $e(t) = \theta^d(t) - \theta(t)$

- Taking the Laplace transformation gives:

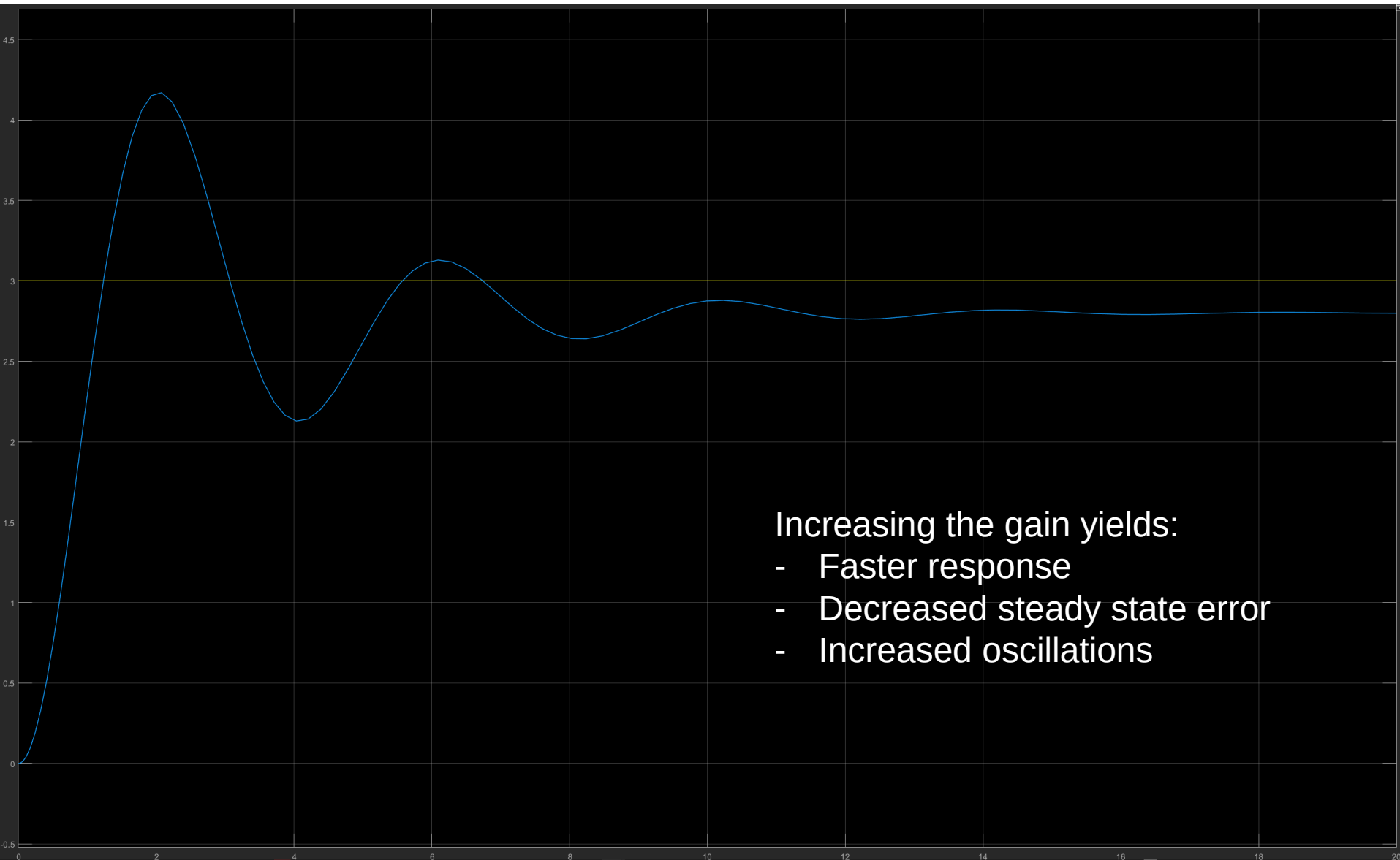
$$U(s) = K_p E(s)$$

- Adding this controller to our system gives the following closed-loop system





$J = 1$ $B = 0.7$ $D = 0.5$ $K_p = 1$



- Increasing the gain yields:
- Faster response
 - Decreased steady state error
 - Increased oscillations

Proportional Derivative (PD) Controller

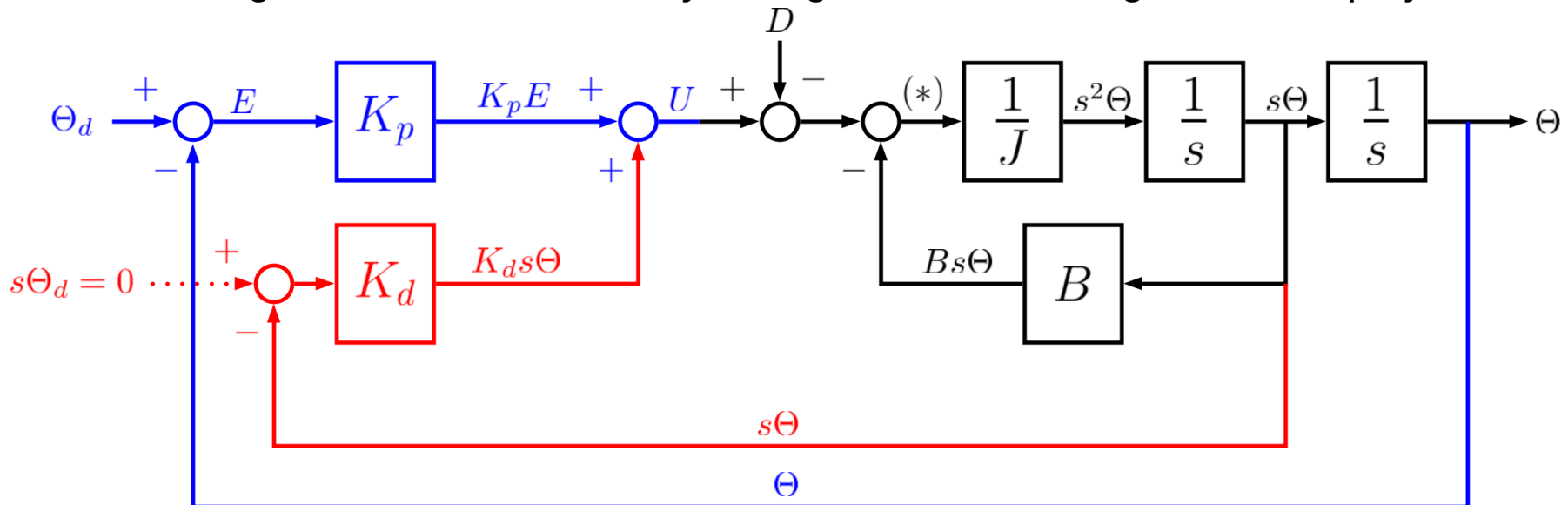
- Control law: $U(t) = K_p e(t) + K_d \dot{e}(t)$

• Where $e(t) = \theta^d(t) - \theta(t)$

- Taking the Laplace transformation gives:

$$U(s) = (K_p + K_d s)E(s)$$

- Adding this controller to our system gives the following closed-loop system:





We see that the Derivative term has a damping effect

Proportional Derivative (PD) Controller

- Recall that this system can be described by:
$$\Theta(s) = \frac{U(s) - D(s)}{Js^2 + Bs}$$

- Where, again, $U(s)$ is:
$$U(s) = (K_p + sK_d)(\Theta^d(s) - \Theta(s))$$

- Combining these gives us:
$$\Theta(s) = \frac{(K_p + sK_d)(\Theta^d(s) - \Theta(s)) - D(s)}{Js^2 + Bs}$$

- Solving for Θ gives:

$$\begin{aligned} (Js^2 + Bs)\Theta(s) + (K_p + sK_d)\Theta(s) &= (K_p + sK_d)\Theta^d(s) - D(s) \\ \Rightarrow (Js^2 + (B + K_d)s + K_p)\Theta(s) &= (K_p + sK_d)\Theta^d(s) - D(s) \\ \Rightarrow \Theta(s) &= \frac{(K_p + sK_d)\Theta^d(s) - D(s)}{Js^2 + (B + K_d)s + K_p} \end{aligned}$$

Proportional Derivative (PD) Controller

- The denominator is the *characteristic polynomial*
- The roots of the characteristic polynomial determine the performance of the system

$$s^2 + \frac{(B + K_d)}{J}s + \frac{K_p}{J} = 0$$

- If we think of the closed-loop system as a damped second order system, this allows us to choose values of K_p and K_d

$$s^2 + 2\zeta\omega s + \omega^2 = 0$$

- Thus K_p and K_d are:

$$K_p = \omega^2 J$$

$$K_d = 2\zeta\omega J - B$$

- A natural choice is $\zeta = 1$ (critically damped)
 - $\zeta < 1$ – underdamped system
 - $\zeta > 1$ – overdamped system

Proportional Derivative (PD) Controller

- Limitations of the PD controller:
 - for illustration, let our desired trajectory be a step input and our disturbance be a constant as well:

$$\Theta^d(s) = \frac{C}{s}, D(s) = \frac{D}{s}$$

- Plugging this into our system description gives:

$$\Theta(s) = \frac{(K_p + sK_d)C - D}{s(Js^2 + (B + K_d)s + K_p)}$$

- For these conditions, what is the steady-state value of the displacement?

$$\theta_{ss} = \lim_{s \rightarrow 0} \frac{s(K_p + sK_d)C - sD}{s(Js^2 + (B + K_d)s + K_p)} = \lim_{s \rightarrow 0} \frac{(K_p + sK_d)C - D}{Js^2 + (B + K_d)s + K_p} = \frac{K_p C - D}{K_p} = C - \frac{D}{K_p}$$

- **Thus the steady state error is $-D/K_p$**
- Therefore to drive the error to zero in the presence of large disturbances, we need large gains... so we turn to another controller

$$\Theta(s) = \frac{(K_d s^2 + K_p s + K_i) \Theta^d(s) - s D(s)}{J s^3 + (B + K_d) s^2 + K_p s + K_i}$$

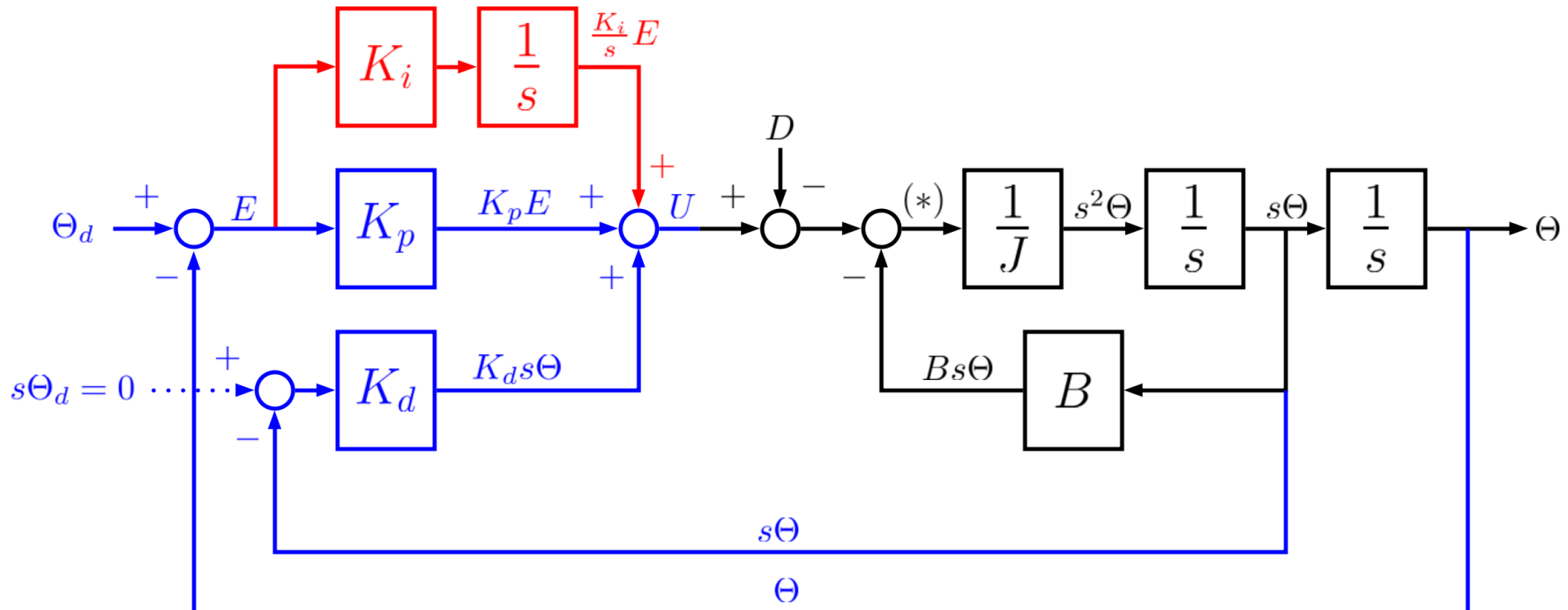
Proportional Integral Derivative (PID) controller

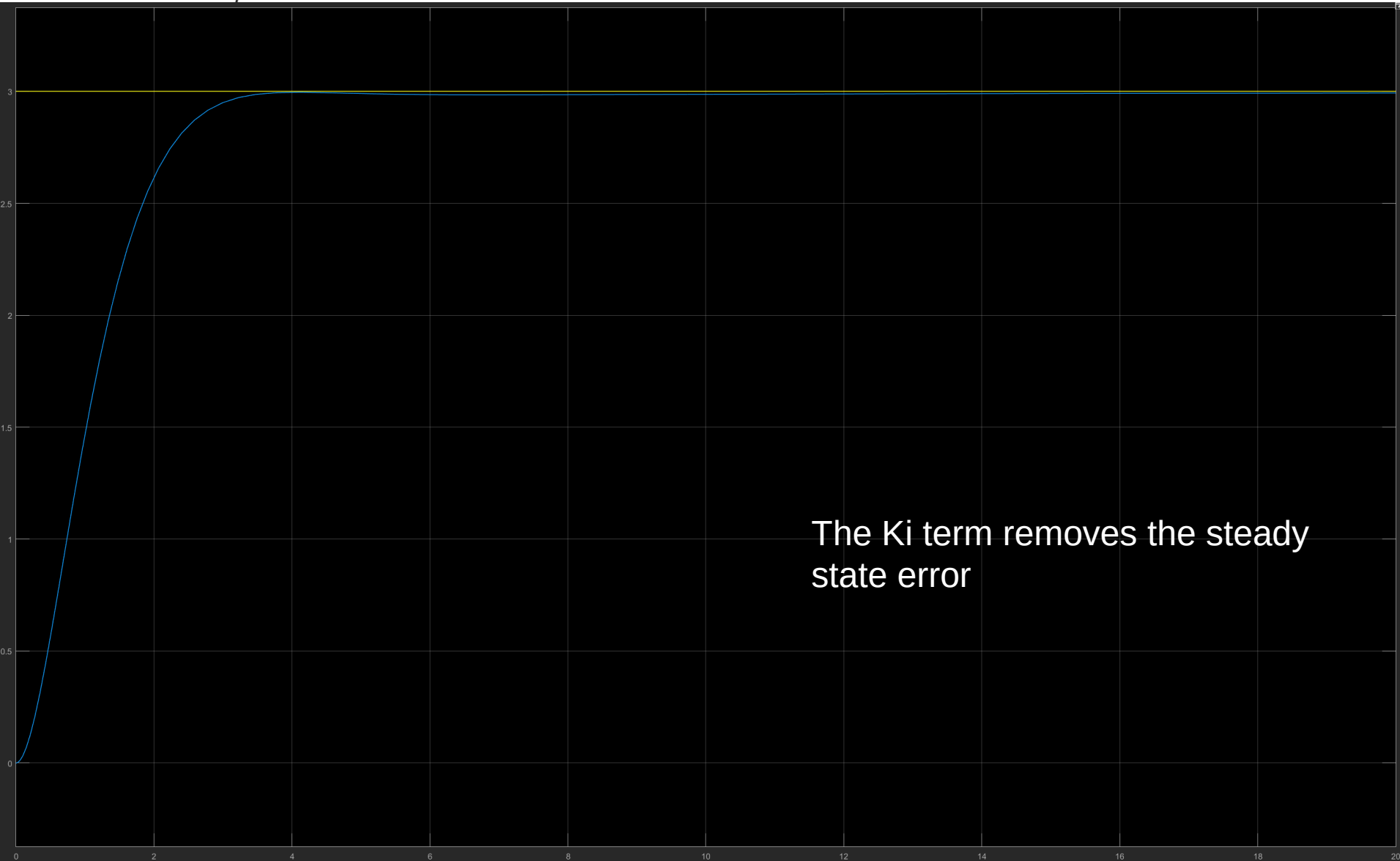
- Control law: $u(t) = K_p e(t) + K_d \dot{e}(t) + K_i \int e(t) dt$

- Taking the Laplace transformation gives:

$$U(s) = \left(K_p + K_d s + \frac{K_i}{s} \right) E(s)$$

- Adding this controller to our system gives the following closed-loop system:





The K_i term removes the steady state error

$$J = 1$$

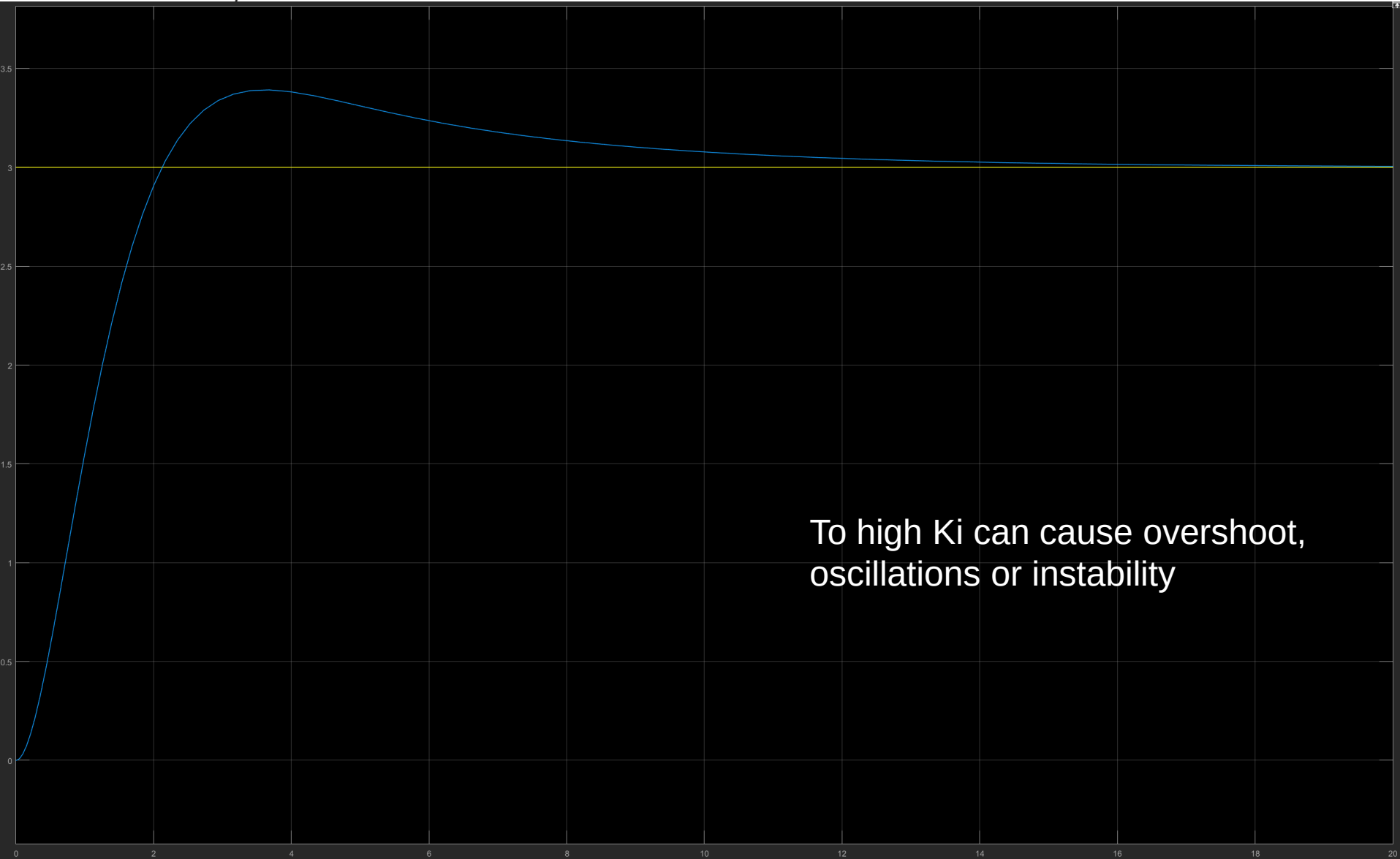
$$B = 0.7$$

$$D = 0.5$$

$$K_p = 2.5$$

$$K_d = 2$$

$$K_i = 0.133$$



To high Ki can cause overshoot,
oscillations or instability

Proportional Integral Derivative (PID) controller

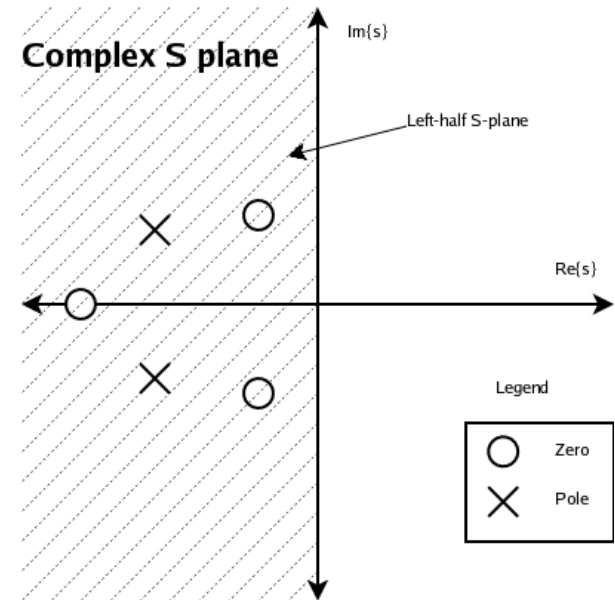
- How to determine PID gains
 1. Set $K_i = 0$ and solve for K_p and K_d
 2. Determine K_i to eliminate steady state error
 - However, we need to be careful of the stability conditions

$$K_i < \frac{(B + K_d)K_p}{J}$$

- In general real world testing we always start with determining K_p
- There are general methods for finding controller gains that could be used (ziegler nichols methods etc.)

Proportional Integral Derivative (PID) controller

- Stability
 - The closed-loop stability of these systems is determined by the roots of the characteristic polynomial
 - If all roots (potentially complex) are in the 'left-half' plane, our system is stable
 - for any bounded input and disturbance
 - A description of how the roots of the characteristic equation change (as a function of controller gains) is very valuable
 - Called the *root locus* (see example slide 36)



Setpoint Controllers – Summary

- **Proportional**
 - A pure proportional controller will have a steady-state error
 - High gain (K_p) will produce a fast system
 - High gain may cause oscillations and may make the system unstable
 - High gain reduces the steady-state error
- **Integral**
 - Removes steady-state error
 - Increasing K_i accelerates the controller
 - High K_i may give oscillations
 - Increasing K_i will increase the settling time
- **Derivative**
 - Larger K_d decreases oscillations
 - Improves stability for low values of K_d
 - May be highly sensitive to noise if one takes the derivative of a noisy error
 - High noise leads to instability



Example - Motor dynamics

- DC motors are ubiquitous in robotics applications
- Here, we develop a transfer function that describes the relationship between the input voltage and the output angular displacement
- First, a physical description of the most common motor: permanent magnet...

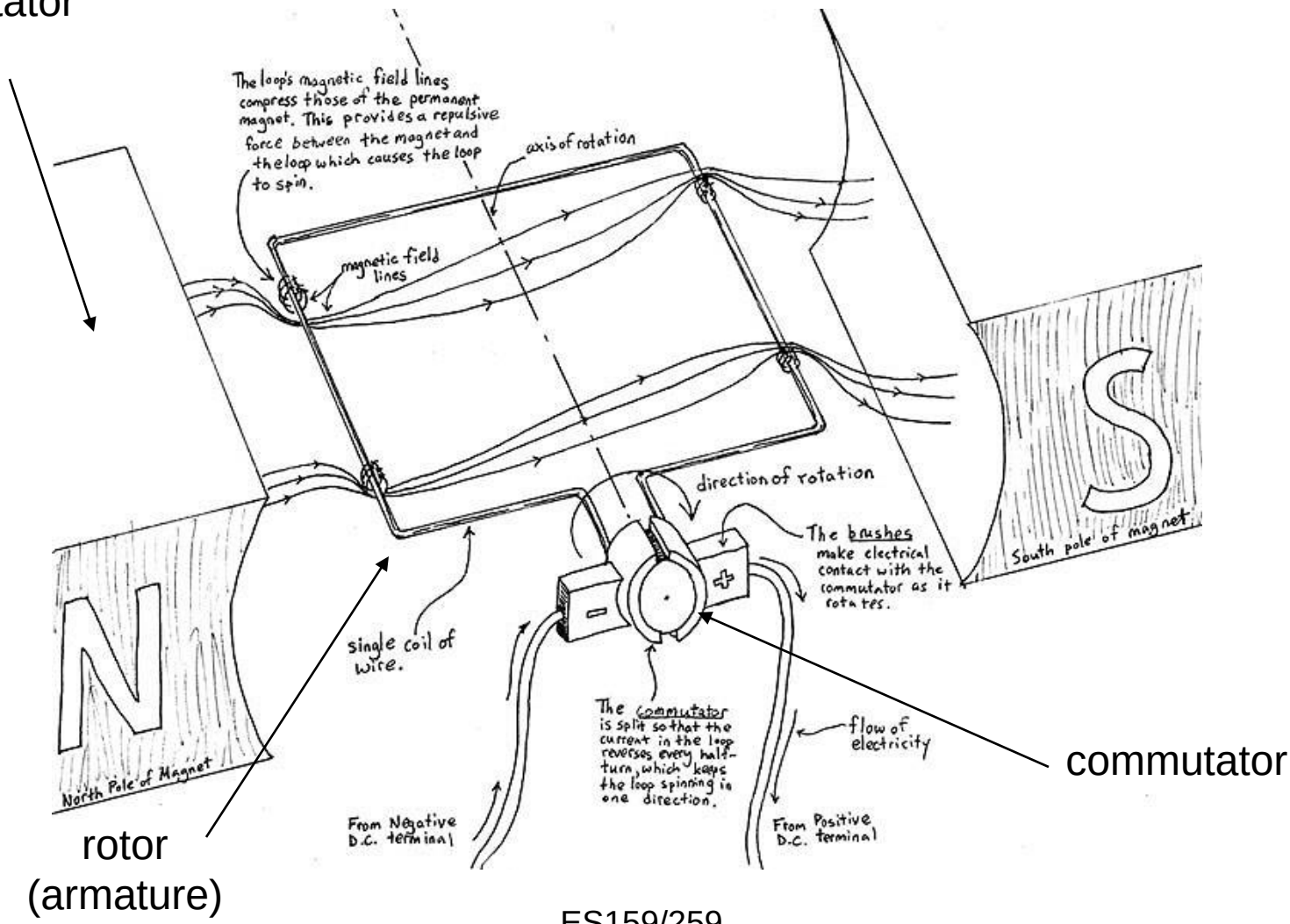
torque on the rotor:

$$\tau_m = K_1 \phi_a$$



Physical instantiation

stator

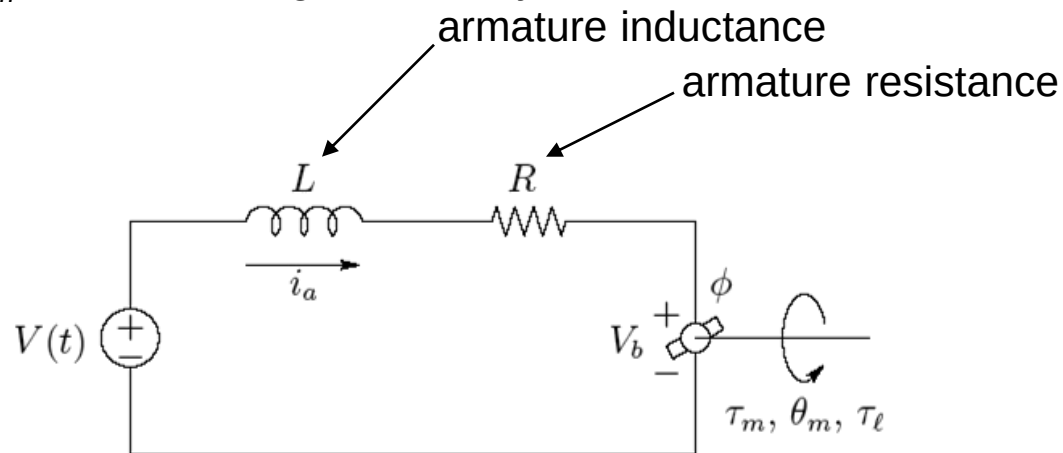


Motor dynamics

- When a conductor moves in a magnetic field, a voltage is generated
 - Called *back EMF*:

$$V_b = K_2 \phi \omega_m$$

- Where ω_m is the rotor angular velocity



$$L \frac{di_a}{dt} + Ri_a = V - V_b$$



Motor dynamics

- Since this is a permanent magnet motor, the magnetic flux is constant, we can write:

$$\tau_m = K_1 \phi i_a = K_m i_a$$

- Similarly:

$$V_b = K_2 \phi \omega_m = K_b \frac{d\theta_m}{dt}$$

torque constant

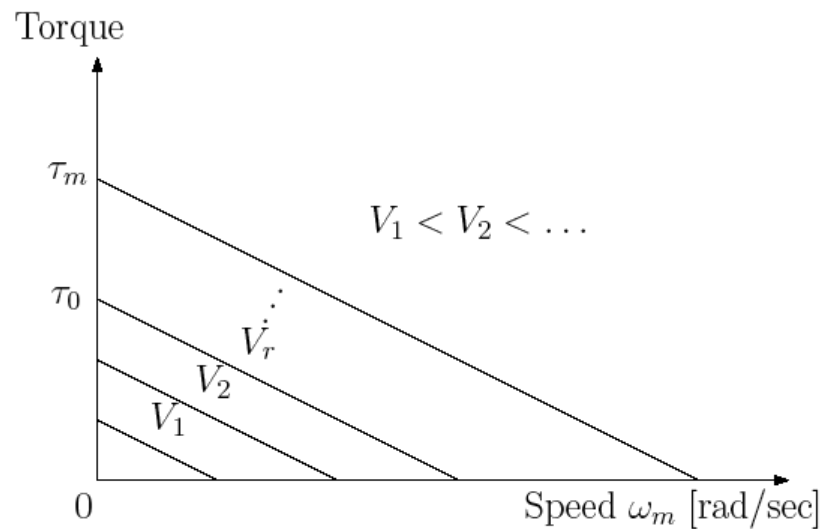
back EMF constant

- K_m and K_b are numerically equivalent, thus there is one constant needed to characterize a motor



Motor dynamics

- This constant is determined from torque-speed curves
 - Remember, torque and displacement are work conjugates

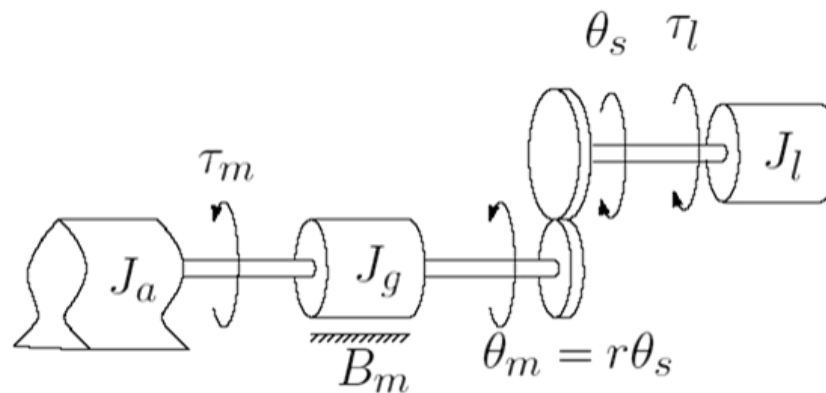


- τ_0 is the *blocked torque*

Single link/joint dynamics

- Now, let's take our motor and connect it to a link
- Between the motor and link there is a gear such that: $\theta_m = r\theta_L$
- Lump the actuator and gear inertias: $J_m = J_a + J_g$
- Now we can write the dynamics of this mechanical system:

$$J_m \frac{d^2\theta_m}{dt^2} + B_m \frac{d\theta_m}{dt} = \tau_m - \frac{\tau_L}{r} = K_m i_a - \frac{\tau_L}{r}$$





Motor dynamics

- Now we have the ODEs describing this system in both the electrical and mechanical domains:

$$L \frac{di_a}{dt} + Ri_a = V - K_b \frac{d\theta_m}{dt}$$

$$J_m \frac{d^2\theta_m}{dt^2} + B_m \frac{d\theta_m}{dt} = K_m i_a - \frac{\tau_L}{r}$$

- In the Laplace domain:

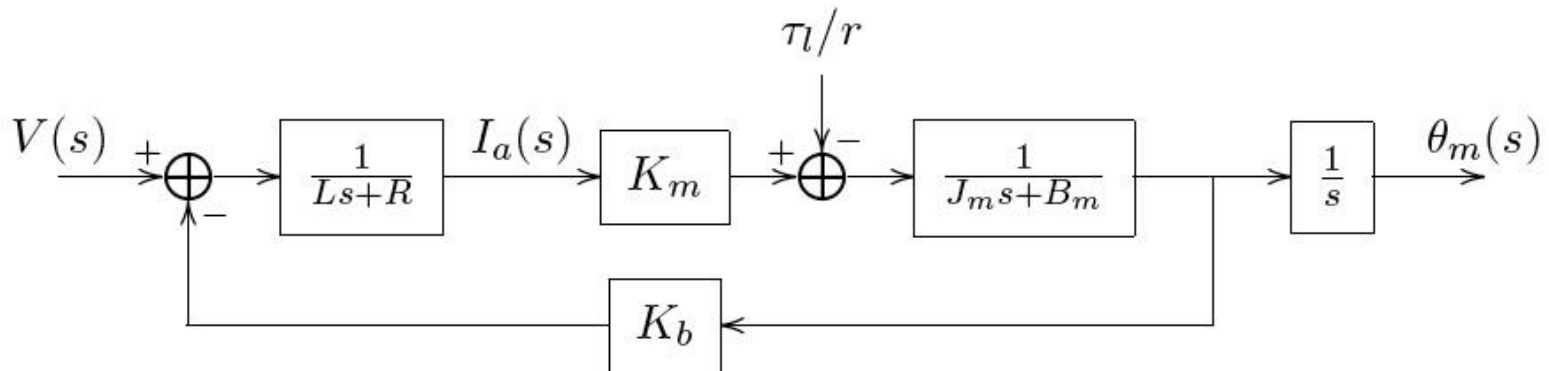
$$(Ls + R)I_a(s) = V(s) - K_b s\Theta_m(s)$$

$$(J_m s^2 + B_m s)\Theta_m(s) = K_m I_a(s) - \frac{\tau_L(s)}{r}$$



Motor dynamics

- These two can be combined to define, for example, the input-output relationship for the input voltage, load torque, and output displacement:





Motor dynamics

- Remember, we want to express the system as a transfer function from the input to the output angular displacement
 - But we have two potential inputs: the load torque and the armature voltage
 - First, assume $\tau_L = 0$ and solve for $\Theta_m(s)$:

$$\frac{(J_m s^2 + B_m s)\Theta_m(s)}{K_m} = I_a(s) \longrightarrow \frac{(Ls + R)(J_m s^2 + B_m s)}{K_m} \Theta_m(s) = V(s) - K_b s \Theta_m(s)$$
$$\longrightarrow \frac{\Theta_m(s)}{V(s)} = \frac{K_m}{s[(Ls + R)(J_m s + B_m) + K_b K_m]}$$



Motor dynamics

- Now consider that $V(s) = 0$ and solve for $\Theta_m(s)$:

$$I_a(s) = \frac{-K_b s \Theta_m(s)}{Ls + R} \longrightarrow (J_m s^2 + B_m s) \Theta_m(s) = \frac{-K_m K_b s \Theta_m(s)}{Ls + R} - \frac{\tau_L(s)}{r}$$
$$\longrightarrow \frac{\Theta_m(s)}{\tau_L(s)} = \frac{-(Ls + R)/r}{s[(Ls + R)(J_m s + B_m) + K_b K_m]}$$

- Note that this is a function of the gear ratio
 - The larger the gear ratio, the less effect external torques have on the angular displacement



Motor dynamics

- In this system there are two ‘time constants’
 - Electrical: L/R
 - Mechanical: J_m/B_m
- For intuitively obvious reasons, the electrical time constant is assumed to be small compared to the mechanical time constant
 - Thus, ignoring electrical time constant will lead to a simpler version of the previous equations:

$$\frac{\Theta_m(s)}{V(s)} = \frac{K_m / R}{s[J_m s + B_m + K_b K_m / R]}$$

$$\frac{\Theta_m(s)}{\tau_L(s)} = \frac{-1/r}{s[J_m s + B_m + K_b K_m / R]}$$



Motor dynamics

- Rewriting these in the time domain gives:

$$\frac{\Theta_m(s)}{V(s)} = \frac{K_m / R}{s[J_m s + B_m + K_b K_m / R]} \longrightarrow J_m \ddot{\theta}_m(t) + (B_m + K_b K_m / R) \dot{\theta}_m(t) = (K_m / R) V(t)$$

$$\frac{\Theta_m(s)}{\tau_L(s)} = \frac{-1/r}{s[J_m s + B_m + K_b K_m / R]} \longrightarrow J_m \ddot{\theta}_m(t) + (B_m + K_b K_m / R) \dot{\theta}_m(t) = -(1/R) \tau_L(t)$$

- By superposition of the solutions of these two linear 2nd order ODEs:

$$\underbrace{J_m}_{J} \ddot{\theta}_m(t) + \underbrace{(B_m + K_b K_m / R)}_B \dot{\theta}_m(t) = \underbrace{(K_m / R) V(t)}_{u(t)} - \underbrace{(1/R) \tau_L(t)}_{d(t)}$$



Motor dynamics

- Therefore, we can write the dynamics of a DC motor attached to a load as:

$$J\ddot{\theta}(t) + B\dot{\theta}(t) = u(t) - d(t)$$

- Note that $u(t)$ is the input and $d(t)$ is the disturbance (e.g. the dynamic coupling from motion of other links)
- To represent this as a transfer function, take the Laplace transform:

