IN3170/4170, Spring 2025

Philipp Häfliger
hafliger@ifi.uio.no

Linear Circuit Analysis

Content

Linear filter and transfer function

Thévenin, Norton, and input- and output resistance/impedance

Bode plots, poles and zeros

Analysis general transconductance amplifier

Dominant pole approximation

Stability

Content

Linear filter and transfer function

Thévenin, Norton, and input- and output resistance/impedance

Bode plots, poles and zeros

Analysis general transconductance amplifier

Dominant pole approximation

Stability

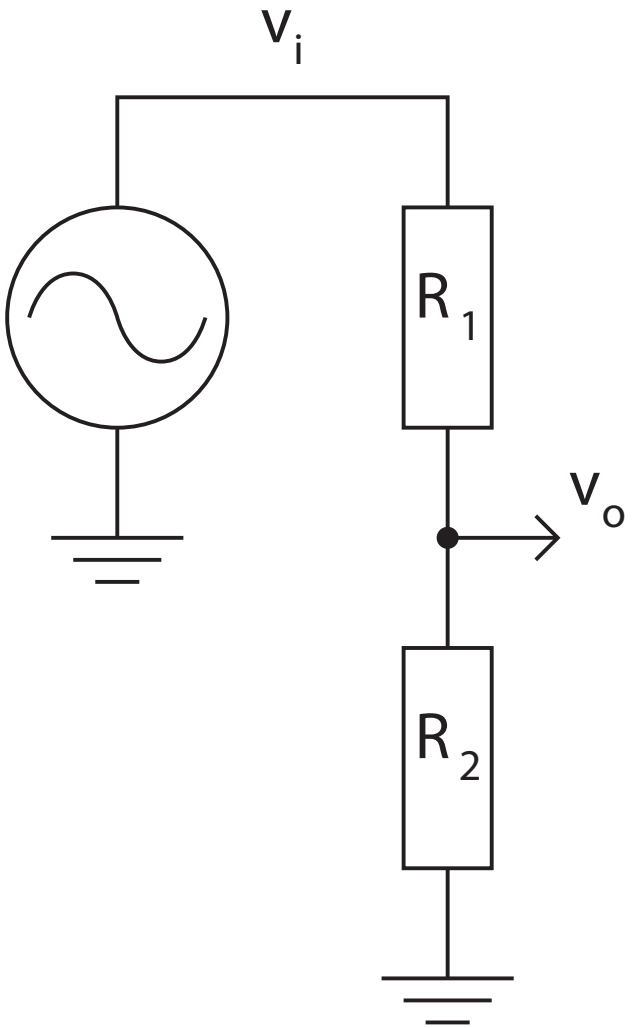
Kirchhoff and Ohm

When analyzing an electronic circuit composed of a network of resistors and signal sources (independent or linearly dependent on other signals in the network), we can simply use Kirchhoff's and Ohm's laws:
Kirchhoff's current law: the sum of all currents into an electric node is zero.
Kirchhoff's voltage law: the sum of all voltages along a electric circuit loop is zero.
Ohm's law:

$$V = I * R \text{ (for now but later also: } V = I * Z)$$

In that case, any signal value in the network (current or voltage) can be deduced knowing a subset of other values in the network.

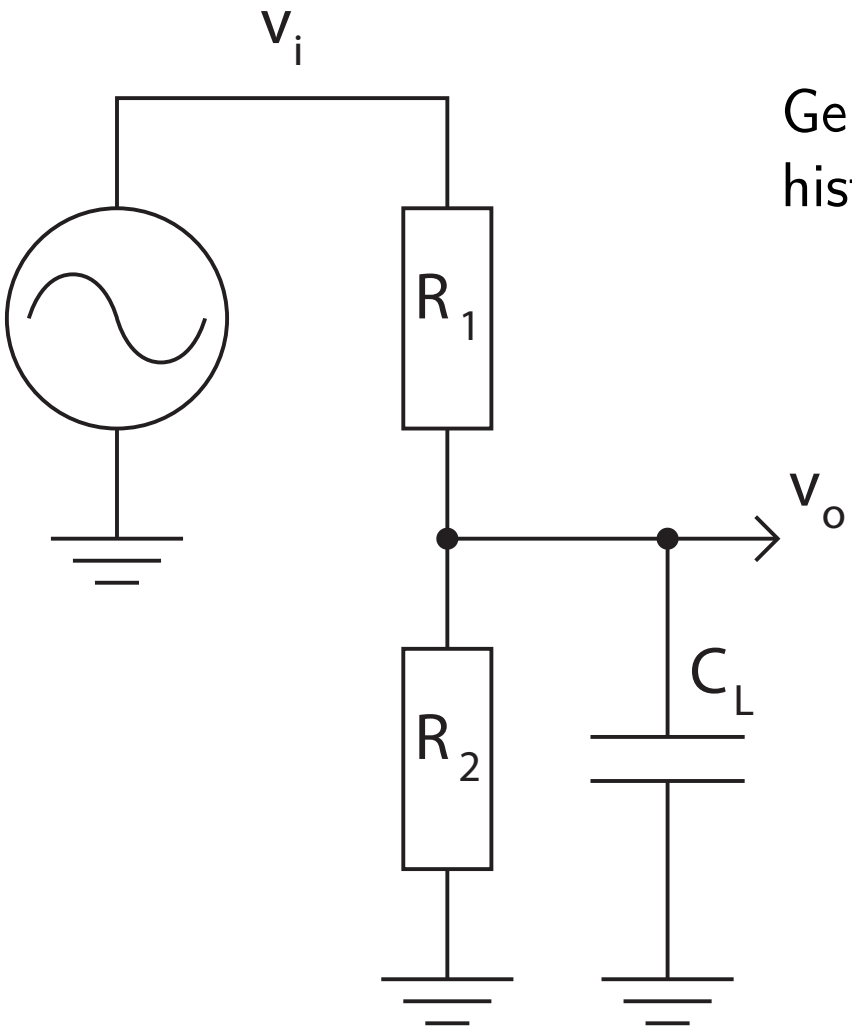
Simple example: Kirchhoff and Ohm



$$v_o(t)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=v_i(t)\frac{1}{R_1}$$

$$v_o(t)=v_i(t)\frac{R_2}{R_1+R_2}$$

Circuit analysis challenge: capacitors and inductors



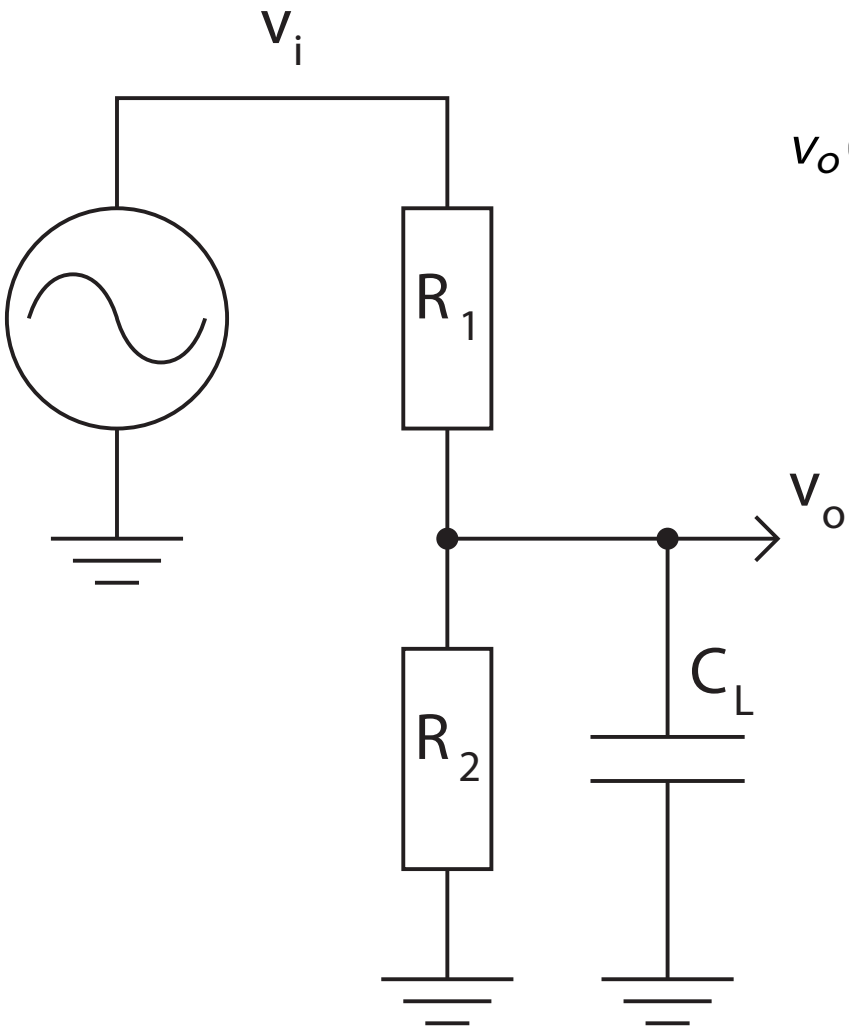
General linear filter: dependent on history and/or signal dynamics

$$I = CV\frac{\partial}{\partial t}$$

$$V = LI\frac{\partial}{\partial t}$$

Need to solve
a differential equation ... :-(

Complex example: Kirchhoff and Ohm



$$v_o(t)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+\dot{v}_o(t)C=v_i(t)\frac{1}{R_1}$$

$$v_o(t)=?$$

Help is on the way: linear circuit analysis (1/2)

Basic ideas:

- Any function can be represented by their frequency spectrum, i.e. an integral/sum of sine waves of different frequencies and phases (Fourier transform)
- Pure sine waves through a linear system will undergo a fixed change of amplitude and phase but maintain their frequency.
- Thus, any function can be decomposed into its frequency spectrum, each pure sine can be sendt through the linear system, and the output function can be reassembled from its frequency components.

Help is on the way: linear circuit analysis (2/2)

Basic ideas:

- Sines can be conveniently represented by the polar form of complex numbers, where the sum of phase φ and sine frequency multiplied with time ωt is the angle and the amplitude the magnitude of the vector in Re, Im space. Th signal is the projection onto the Im-axis.
- In an electric circuit the sine wave signal can undergo addition, multiplication with a scalar, and derivation and integration.
- Addition of sines is equivalent to vector addition to their complex number representation, and so is multiplication with a scalar
- derivation is a 90^o phase shift and multiplication with ω, corresponding to a multiplication with $j\omega$, and integration is a multiplication with $\frac{1}{j\omega}$

Definition: linear system

Definition: A **linear system** is an operator h that transforms an input signal (i.e. a function $x(t)$ of time t) into an output signal $h(x(t))$ such that $h(x(t)) + h(y(t)) = h(x(t) + y(t))$ and $sh(x(t)) = h(sx(t))$, for any scalar value s.

So derivatives qualify (as do general integrals):

$$\begin{aligned} I &= [C(V_1(t) + V_2(t))] \frac{\partial}{\partial t} \\ &= C\left[(V_1(t) + V_2(t))\frac{\partial}{\partial t}\right] \\ &= C\left[V_1(t)\frac{\partial}{\partial t} + V_2(t)\frac{\partial}{\partial t}\right] \end{aligned}$$

.. and thus any I/O relationship between two nodes in an electric network including resistors and 'artificial' linear elements, as well as capacitors and inductors is a linear system.

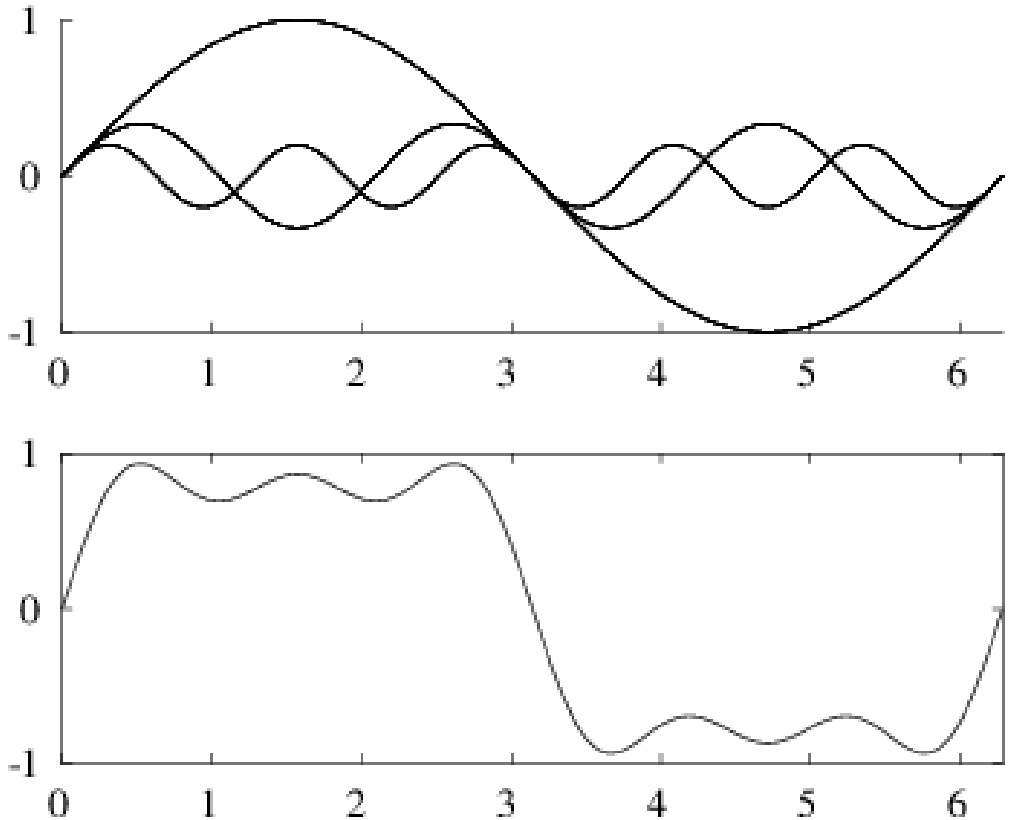
Assuming the Fourier Transform works...

We are not proofing it here, but just stipulate: any function can be represented as the sum/integral of sine waves at different phase shift. The Fourrier transform is the method of decomposing a function of time into such a sum/integral over frequency, i.e. the Fourrier transform computes the *frequency spectrum* of a function and the inverse Fourrier transform reassembles the function from its spectrum.

Thus by definition of a linear system, we can decompose a function into sine waves, send them individually through a linear system, and sum the results for each frequency at the output. So if we know what the linear system does to any sine wave, we have fully described the linear system.

Illustration Fourier

Making a square wave from a sum of sines:



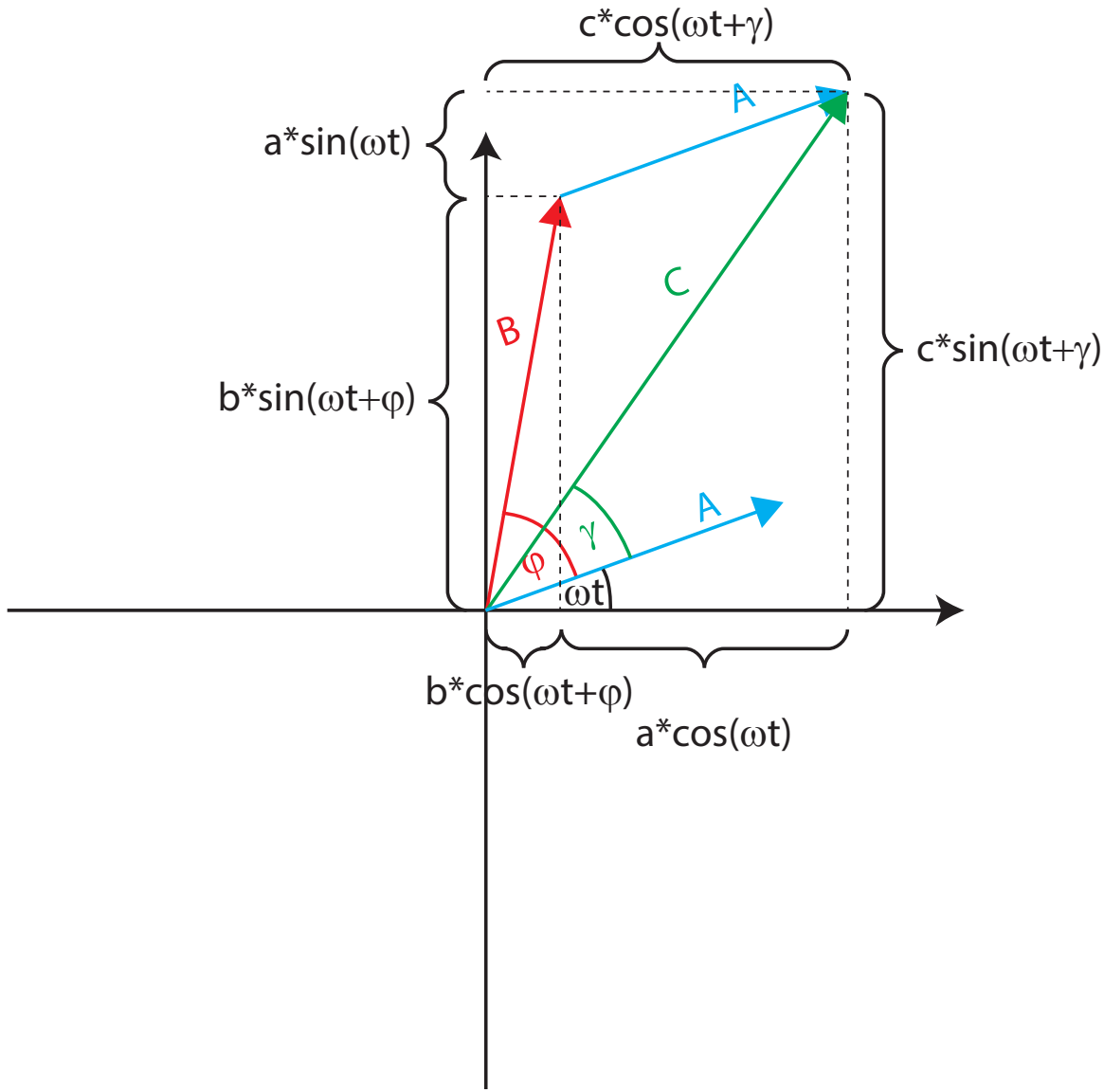
What happens to a sine wave: derivation

$$V := \sin(\omega t + \varphi)$$

$$I = CV\frac{\partial}{\partial t} = C\omega\cos(\omega t + \varphi) = C\omega\sin(\omega t + \varphi + \frac{\pi}{2})$$

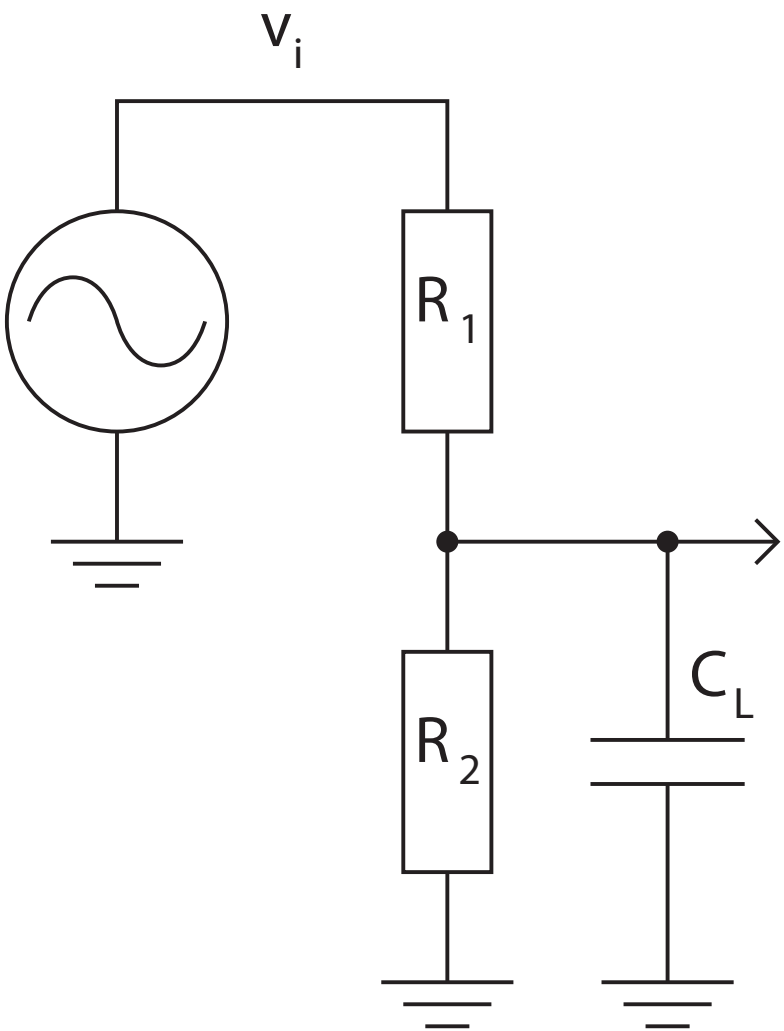
Note: result has the same frequency!

What happens to a sine wave: scalar multiplication, addition



Note: result has the same frequency!

Complex example: Kirchhoff and Ohm



$$v_o(t)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+\dot{v}_o(t)C = v_i(t)\frac{1}{R_1}$$

$$v_i(\omega) := (\mathcal{F}v_i(t))(\omega)$$

$$v_o(\omega)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+v_o(\omega)j\omega C = v_i(\omega)\frac{1}{R_1}$$

$v_o(\omega)$

$$= v_i(\omega)\frac{\frac{1}{R_1}}{\frac{1}{R_1}+\frac{1}{R_2}+j\omega C}$$

$$= v_i(\omega)\frac{R_2}{R_1+R_2}\frac{1}{1+\frac{R_1R_2}{R_1+R_2}j\omega C}$$

Content

Linear filter and transfer function

Thévenin, Norton, and input- and output resistance/impedance

Bode plots, poles and zeros

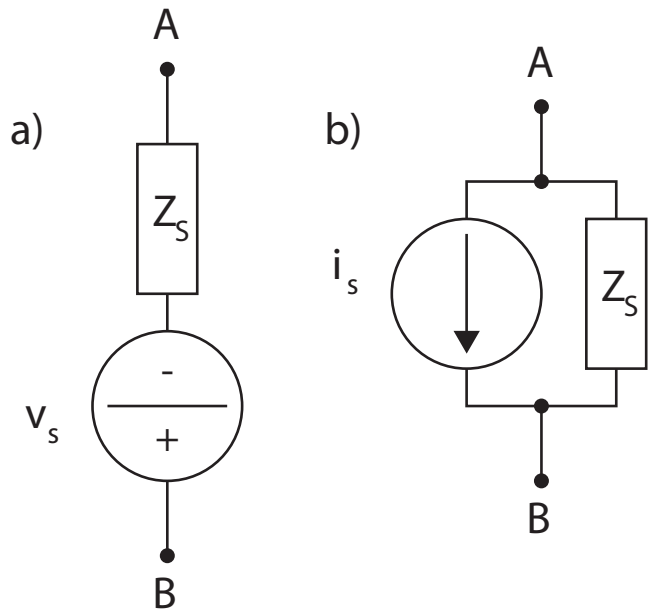
Analysis general transconductance amplifier

Dominant pole approximation

Stability

Thévenin, Norton, and input- and output resistance/impedance

In linear electronics analysis (sub-)networks can be replaced by a single equivalent resistance/impedance, and (provided the presence of signal-, power-, biasing- sources in said network) with either a voltage source in series or a current source in parallel:



a) is called Thévenin form and b) Norton form. Z_S, v_s, i_s can be found experimentally or analytically. a) and b) are equivalent where $v_s = i_s * Z_S$. If node B is Gnd, Z_S is also referred to as input- or output impedance, or the impedance looking into the network from node A.

Content

Linear filter and transfer function

Thévenin, Norton, and input- and output resistance/impedance

Bode plots, poles and zeros

Analysis general transconductance amplifier

Dominant pole approximation

Stability

Root form of a transfer function

Solving a linear electronic network's I/O relationships $v_o = Hv_i$, H can be written as the ratio of two polygons in $s := j\omega$, i.e.:

$$H = \frac{A}{B} = \frac{a_0 + a_1 s + a_2 s^2 ... a_n s^n}{b_0 + b_1 s + b_2 s^2 ... b_m s^m}$$

The selvsame in root form:

$$H = \frac{a_0}{b_0} \frac{(1 + \frac{s}{\omega_{z_1}})...(1 + \frac{s}{\omega_{z_n}})}{(1 + \frac{s}{\omega_{p_1}})...(1 + \frac{s}{\omega_{p_m}})}$$

Where the **zero**- and **pole** frequencies are ordered by magnitude $|\omega_{z_1}| \leq |\omega_{z_2}| \leq ... \leq |\omega_{z_n}|$ and $|\omega_{p_1}| \leq |\omega_{p_2}| \leq ... \leq |\omega_{p_m}|$.

Decibel [dB]

A logarithmic unit to express relative gain in a transfer function
Gain in amplitude where +20dB correspond to a multiplication with 10:

$$A[\text{dB}] = 20 \log_{10} A[\text{lin}]$$

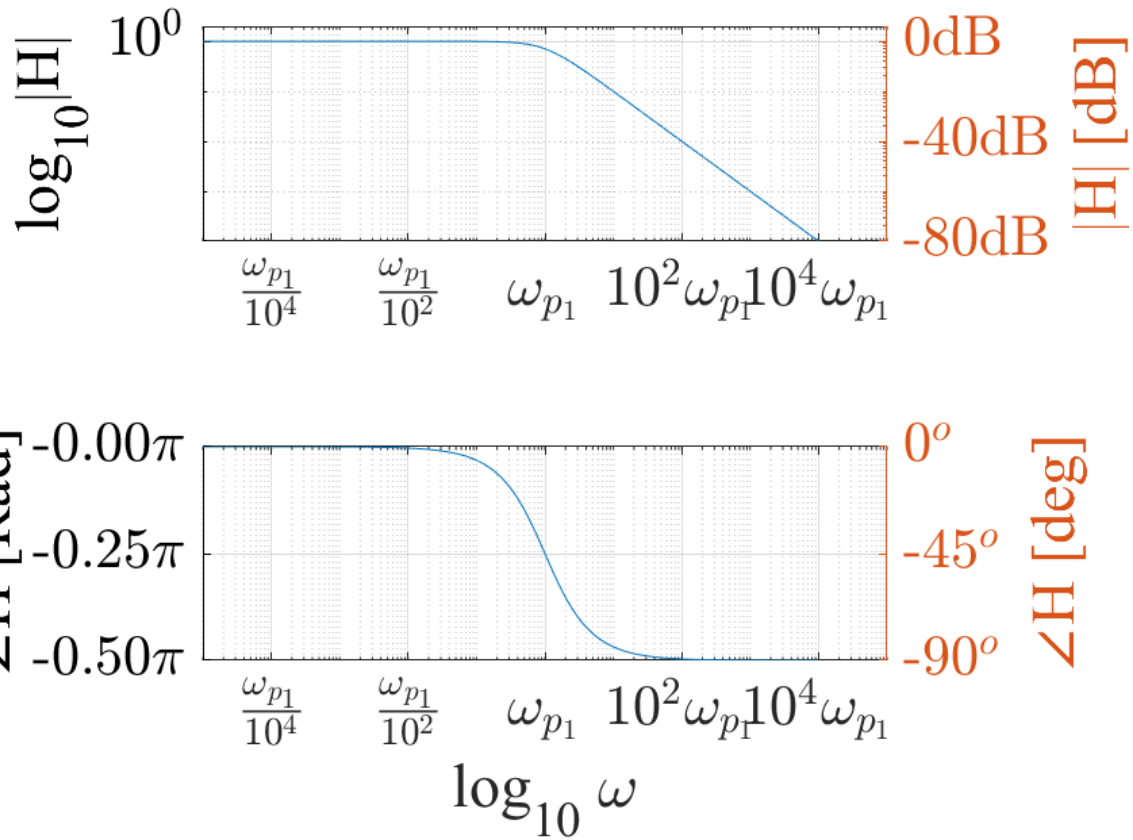
Gain in power (which is proportional to the gain in amplitude squared, for instance $P = VI = \frac{V^2}{R} = I^2 R$):

$$A^2[\text{dB}] = 10 \log_{10} A^2[\text{lin}]$$

Single Real Pole Effect (Bode Plot)

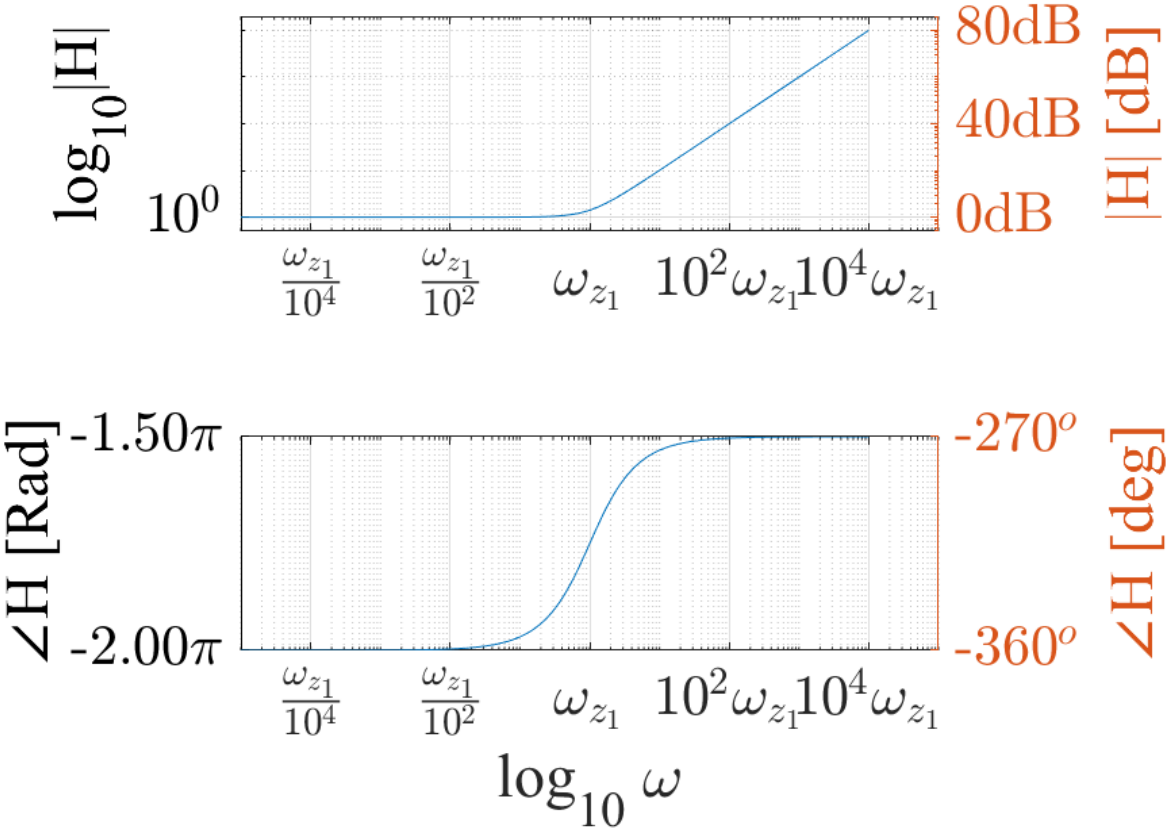
Note: transfer function evaluated for $s = j\omega$ (the imaginary axis), i.e. does not 'hit' that real pole.

$$\frac{1}{1+\frac{s}{\omega_{p_i}}} \begin{cases} \approx 1 & \text{if } |s| << \omega_{p_i} \\ = \frac{1}{\sqrt{2}} e^{-j\frac{\pi}{4}} & \text{if } |s| = \omega_{p_i} \\ \approx -j\frac{\omega_{p_i}}{\omega} & \text{if } |s| >> \omega_{p_i} \end{cases}$$



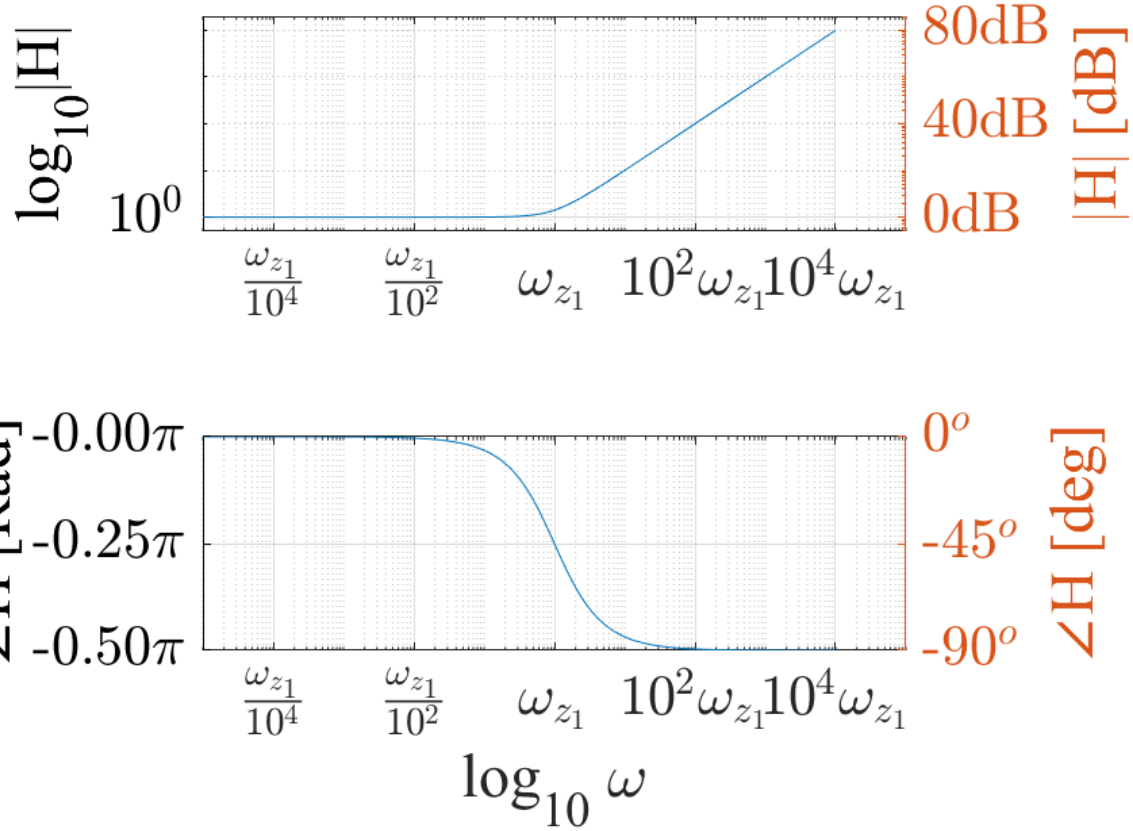
Single Negative Real Zero Effect

$$1 + \frac{s}{\omega_{z_i}} \begin{cases} \approx 1 & \text{if } |s| << \omega_{z_i} \\ = \sqrt{2} e^{j\frac{\pi}{4}} & \text{if } |s| = \omega_{z_i} \\ \approx j\frac{\omega}{\omega_{z_i}} & \text{if } |s| >> \omega_{z_i} \end{cases}$$



Single Positive Real Zero Effect

$$1 - \frac{s}{\omega_{z_i}} \begin{cases} \approx 1 & \text{if } |s| << \omega_{z_i} \\ = \sqrt{2} e^{-j\frac{\pi}{4}} & \text{if } |s| = \omega_{z_i} \\ \approx -j\frac{\omega}{\omega_{z_i}} & \text{if } |s| >> \omega_{z_i} \end{cases}$$



Rules to Draw Bode Plots from Root Form Transfer Functions

1. In the magnitude plot there is a convex knee at each real pole frequency changing the slope by -20dB/decade, and concave knee for each real zero changing the slope by +20dB/decade.
2. In the phase plot there is a step down of $-\frac{\pi}{2}$ for each negative real pole and each positive real zero and a step up by $\frac{\pi}{2}$ for each negative real zero.
3. The curve is flat if it has an equal number of poles and zeros at lower frequencies (so for example with no zero or pole at point 0 the curve starts flat with the magnitude being $\frac{a_o}{b_o}$.)

Content

Linear filter and transfer function

Thévenin, Norton, and input- and output resistance/impedance

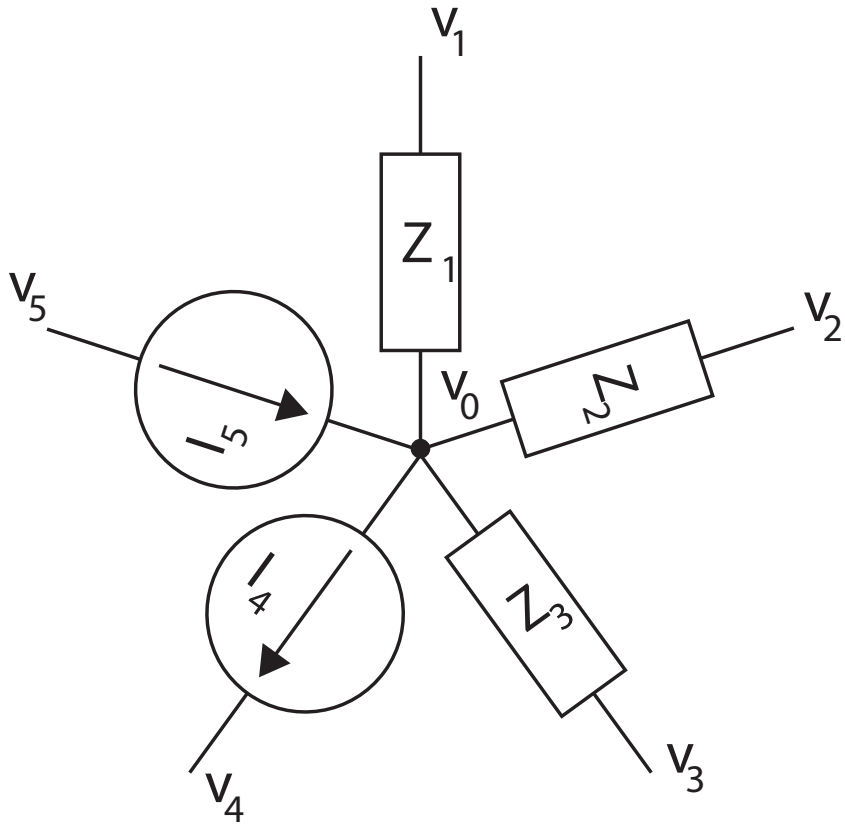
Bode plots, poles and zeros

Analysis general transconductance amplifier

Dominant pole approximation

Stability

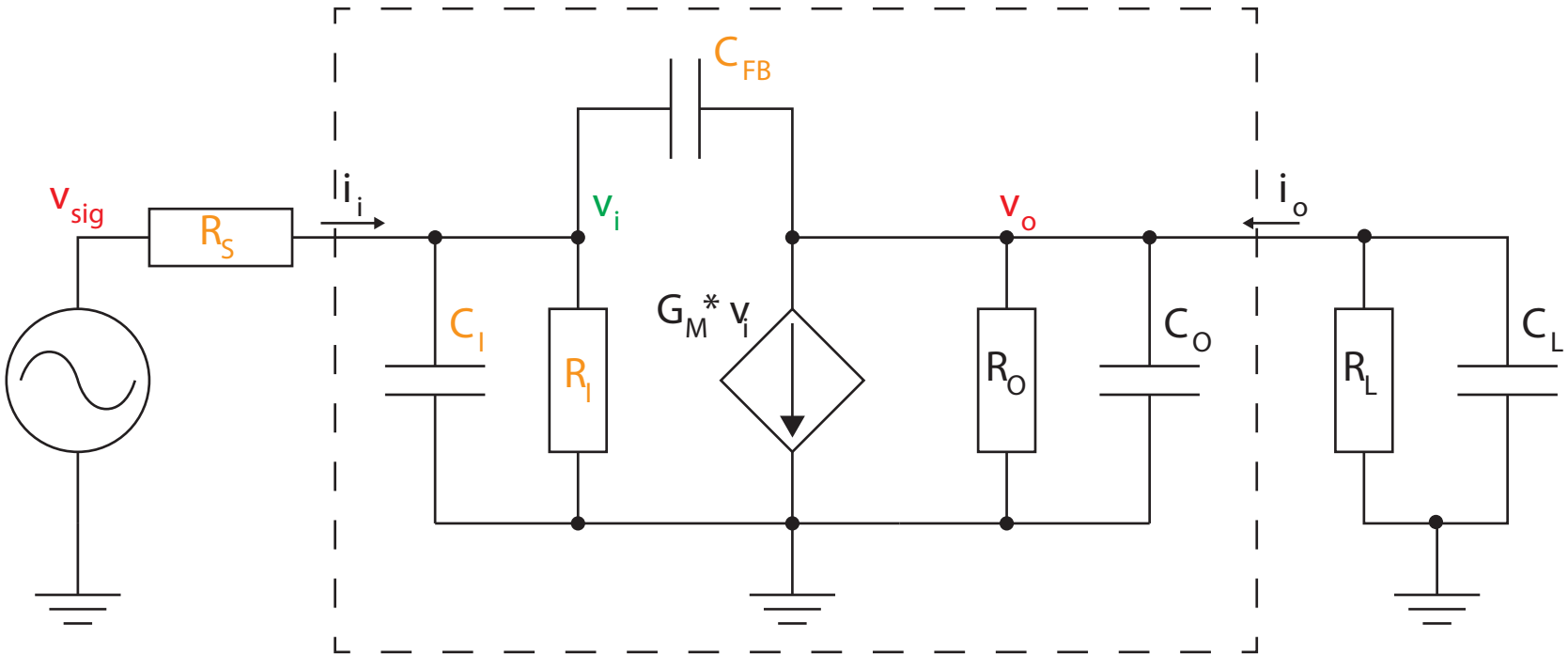
Setting up Kirchhoff equations



$$v_0\left(\frac{1}{Z_1}+\frac{1}{Z_2}+\frac{1}{Z_3}\right)+I_4=v_1\frac{1}{Z_1}+v_2\frac{1}{Z_2}+v_3\frac{1}{Z_3}+I_5$$

Analysis of a general 2nd order transconductance amplifier

A useful general model suited for the aforementioned common source amp and some others.



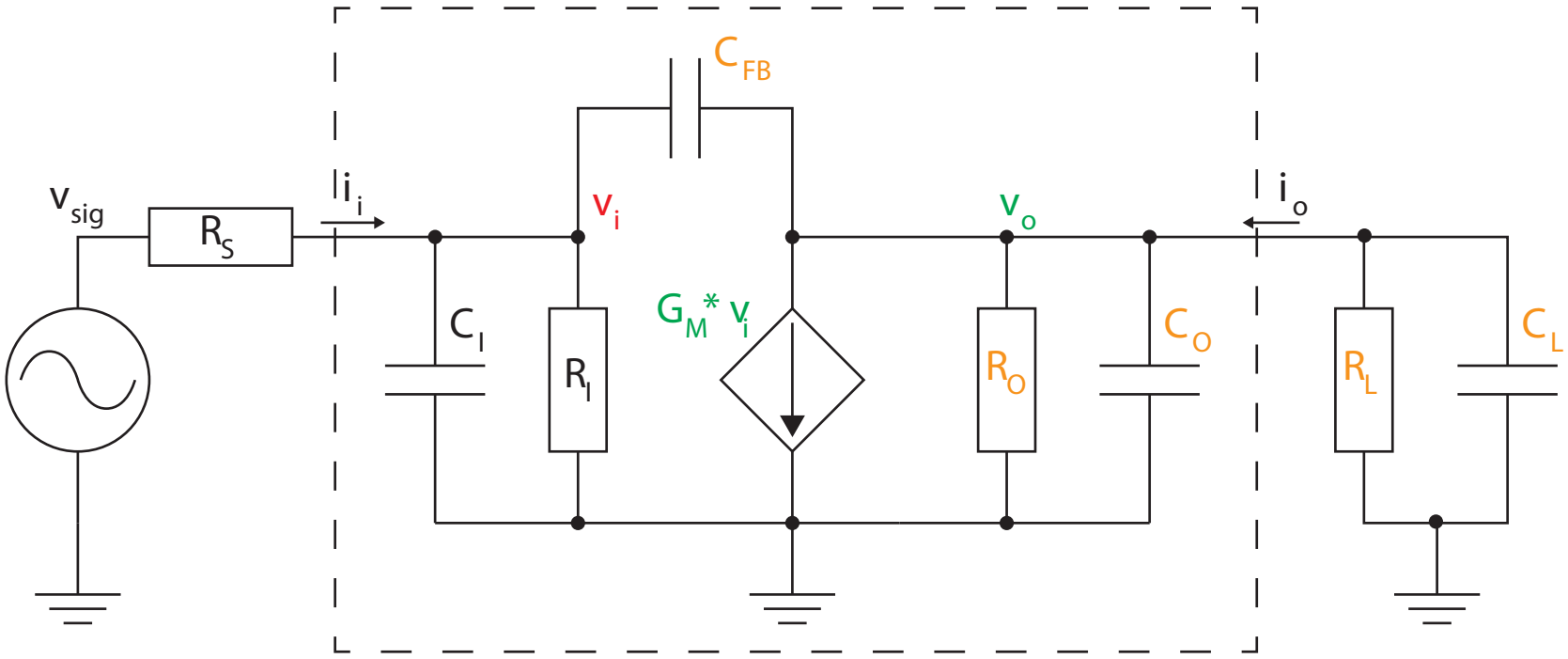
Kirchhoff equation for node v_i:

$$v_i \left(\frac{1}{R_I^*} + s\left(C_{FB} + C_I\right) \right) = v_{sig} \frac{1}{R_{SIG}} + v_o s C_{FB}$$

Where:

$$R_I^* := R_I || R_{SIG}$$

Analysis of a general 2nd order transconductance amplifier



Kirchhoff equation for node v_o:

$$v_o \left(\frac{1}{R_L^*} + s\left(C_{FB} + C_L^*\right) \right) + v_i G_M = v_i s C_{FB}$$

Where:

$$R_L^* := R_O || R_L \ , \ C_L^* := C_O || C_L$$

Analysis of a general 2nd order transconductance amplifier

Transfer function result:

$$\frac{v_o}{v_{sig}} = -G_M R_L^* \frac{R_I}{R_I + R_{SIG}} \quad \frac{(1 - s\frac{C_{FB}}{G_M})}{1 + sB + s^2 A}$$

Where:

$$B = R_L^*(C_{FB} + C_L^*) + R_I^*(C_I + C_{FB}(1 + G_M R_L^*))$$
$$A = R_I^* R_L^* \left[(C_{FB} + C_L^*)(C_{FB} + C_I) - C_{FB}^2\right]$$

Special case when $C_{FB} = 0$:

$$\frac{v_o}{v_{sig}} = -G_M R_L^* \frac{R_I}{R_I + R_{SIG}} \quad \frac{1}{(1 + R_I^* C_I)(1 + R_L^* C_L^*)}$$

i.e. simply two 1st order filters in series with poles at frequencies $\frac{1}{R_I^* C_I}$ and $\frac{1}{R_L^* C_L^*}$

Content

Linear filter and transfer function

Thévenin, Norton, and input- and output resistance/impedance

Bode plots, poles and zeros

Analysis general transconductance amplifier

Dominant pole approximation

Stability

Dominant Pole Approximation

Condition/definition for p_1 being a 'dominant' pole:

$$\forall\{i > 1, j > 0 : (\omega_{pi} > 4\omega_{p1}) \wedge (\omega_{zj} > 4\omega_{p1})\} \Leftrightarrow \omega_{-3\text{dB}} \approx \omega_{p_1}$$

If that is the case, life gets easier. And even better if:

$$\forall\{i > 1, j > 0 : (\omega_{pi} > A_{\text{DC}}\omega_{p1}) \wedge (\omega_{zj} > A_{\text{DC}}\omega_{p1})\} \Leftrightarrow \omega_T \approx A_{\text{DC}}\omega_{p1}$$

Where ω_T is defined as the frequency where the gain is reduced to being 1:

$$|A(j\omega_T)| = 1$$

Spotting a dominant pole in the transfer function

With this denominator of the transfer function

$$b_0 + b_1 s + b_2 s^2 ... b_m s^m$$

$$\forall \{ i > 1 : |b_i| << |b_1| \} \Leftrightarrow \frac{1}{\omega_{p_1}} \approx \frac{b_1}{b_o} \tag{1}$$

Second order example:

$$1 + \left(\frac{1}{\omega_{p_1}} + \frac{1}{\omega_{p_2}} \right) s + \frac{1}{\omega_{p_1} \omega_{p_2}} s^2 \tag{2}$$

So if $\omega_{p_2} >> \omega_{p_1} > 1$ then the term $\frac{1}{\omega_{p_1}} + \frac{1}{\omega_{p_2}}$ is dominated by $\approx \frac{1}{\omega_{p_1}}$ and $\frac{1}{\omega_{p_1} \omega_{p_2}} << \left(\frac{1}{\omega_{p_1}} + \frac{1}{\omega_{p_2}} \right)$.

Simplifying the transconductance amplifier transfer function in special cases

- No feedback capacitor
- Assuming dominant pole at output
- Assuming dominant dple at input

No FB Capacitor

Assuming $C_{FB} = 0$ the TransAmp transfer function becomes:

$$\frac{v_o}{v_{sig}} = -G_M R_L^* \frac{R_I}{R_I + R_{SIG}} \quad \frac{1}{(1 + R_I^* C_I)(1 + R_L^* C_L^*)}$$

This is then simply two independent 1^{st} order filters in series.

Assuming dominant pole at output

(for example by assuming $R_{SIG} = 0$ and consequently $v_i = v_{sig}$)

$$\frac{v_o}{v_{sig}} = -G_M R_L^* \frac{1 - s\frac{C_{FB}}{G_M}}{1 + s\left(R_L^*(C_{FB} + C_L^*)\right)}$$

So one pole and one zero:

$$\omega_{p_1} = \frac{1}{R_L^*(C_{FB} + C_L^*)}$$

$$\omega_{z_1} = \frac{G_M}{C_{FB}}$$

$$\omega_T = |A_{DC}|\omega_{p_1} = \frac{G_M}{C_{FB} + C_L^*}$$

$$\omega_{p_1} < \omega_T < \omega_{z_1}$$

(For the first comparison we need $A_{DC} > 1$.)

So the zero after the unity gain frequency does usually not have an important impact.

Assuming dominant pole at input

Assuming $v_o = -G_M R_L^* v_i$ and only v_i to be affected by frequency, and thus Kirchhoff for output node:

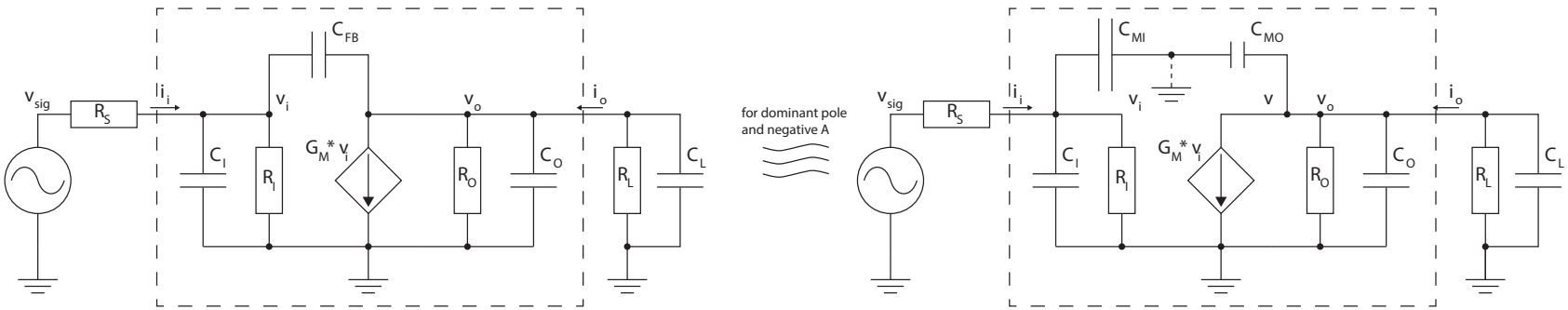
$$v_o \frac{1}{R_L^*} + v_i G_M = 0$$

and transfer function:

$$\frac{v_o}{v_{sig}} = -G_M R_L^* \frac{R_I}{R_{SIG} + R_I} \frac{1}{1 + sR_I^* \left(C_I + C_{FB}(1 + G_M R_L^*)\right)}$$

Virtually enhancing C_{FB} with the 'Miller term' $(1 + G_M R_L^*)$.

Equivalent schematics explaining the feed back Miller effect



$$C_{MI} = (1 + A_{DC})C_{FB} = (1 + G_M R_L^*)C_{FB}$$

$$\left(C_{MO} = \left(\frac{1}{A_{DC}} + 1 \right) C_{FB} \right)$$

Content

Linear filter and transfer function

Thévenin, Norton, and input- and output resistance/impedance

Bode plots, poles and zeros

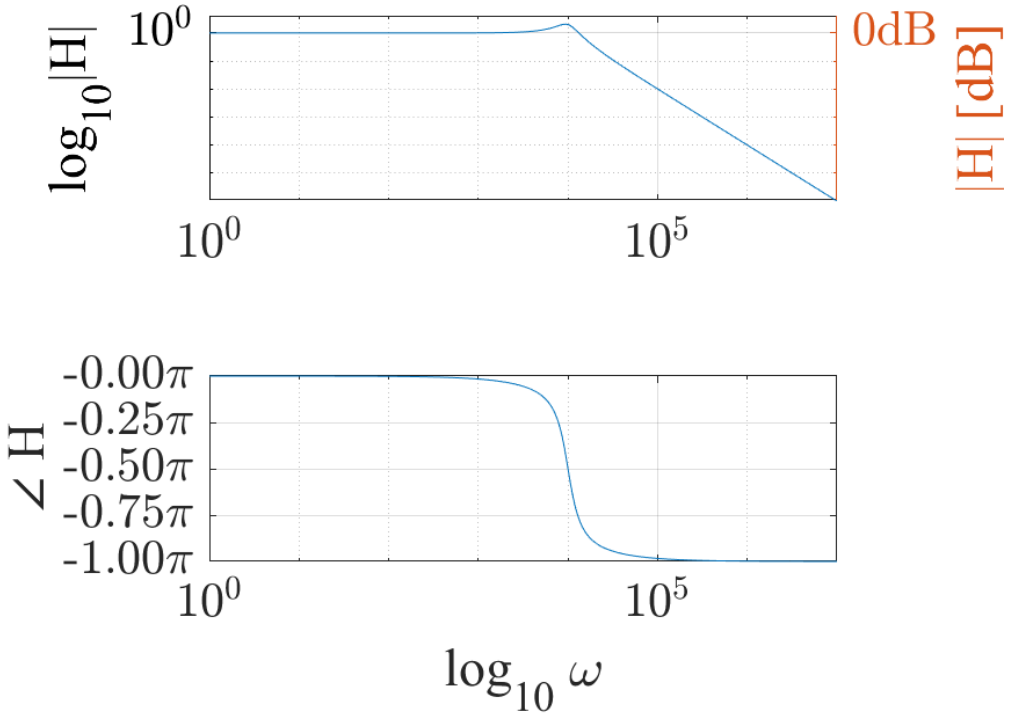
Analysis general transconductance amplifier

Dominant pole approximation

Stability

Stability

Feedback loops can cause resonance or even become unstable. 2^{nd} or higher order lowpass transfer functions can show resonance/instability when they do have *complex* poles.



Two methods to predict resonance/unstable behaviour:

- Q-factor
- Phase margin (PM)

Q-factor

The Q-factor indicates the amount of resonance of 2^{nd} order LP filters. It is defined by rewriting the denominator of the second order transfer function as follows:

$$H(s) = \ldots \frac{\ldots}{1 + \frac{1}{\omega_0 Q} s + \frac{1}{\omega_0^2} s^2}$$

Solving: $0 = 1 + \frac{1}{\omega_0 Q} s + \frac{1}{\omega_0^2} s^2$

$$p_1, p_2 = \frac{-\frac{1}{\omega_0 Q} \pm \sqrt{\frac{1}{\omega_0^2 Q^2} - 4\frac{1}{\omega_0^2}}}{2\frac{1}{\omega_0^2}} = \frac{-\frac{\omega_0}{Q} \pm \sqrt{\frac{\omega_0^2}{Q^2} - 4\omega_0^2}}{2}$$

$$\frac{1}{Q^2} < 4 \Leftrightarrow |Q| > 0.5$$

In that case p_1 and p_2 will be a conjugate complex pair, and their magnitude will be exactly $|p_{1,2}| = \omega_0$. Rules to draw Bode plots from **real** poles no longer apply! Complications ensue.

Q-factor ≤ 0.5 (1/2)

if

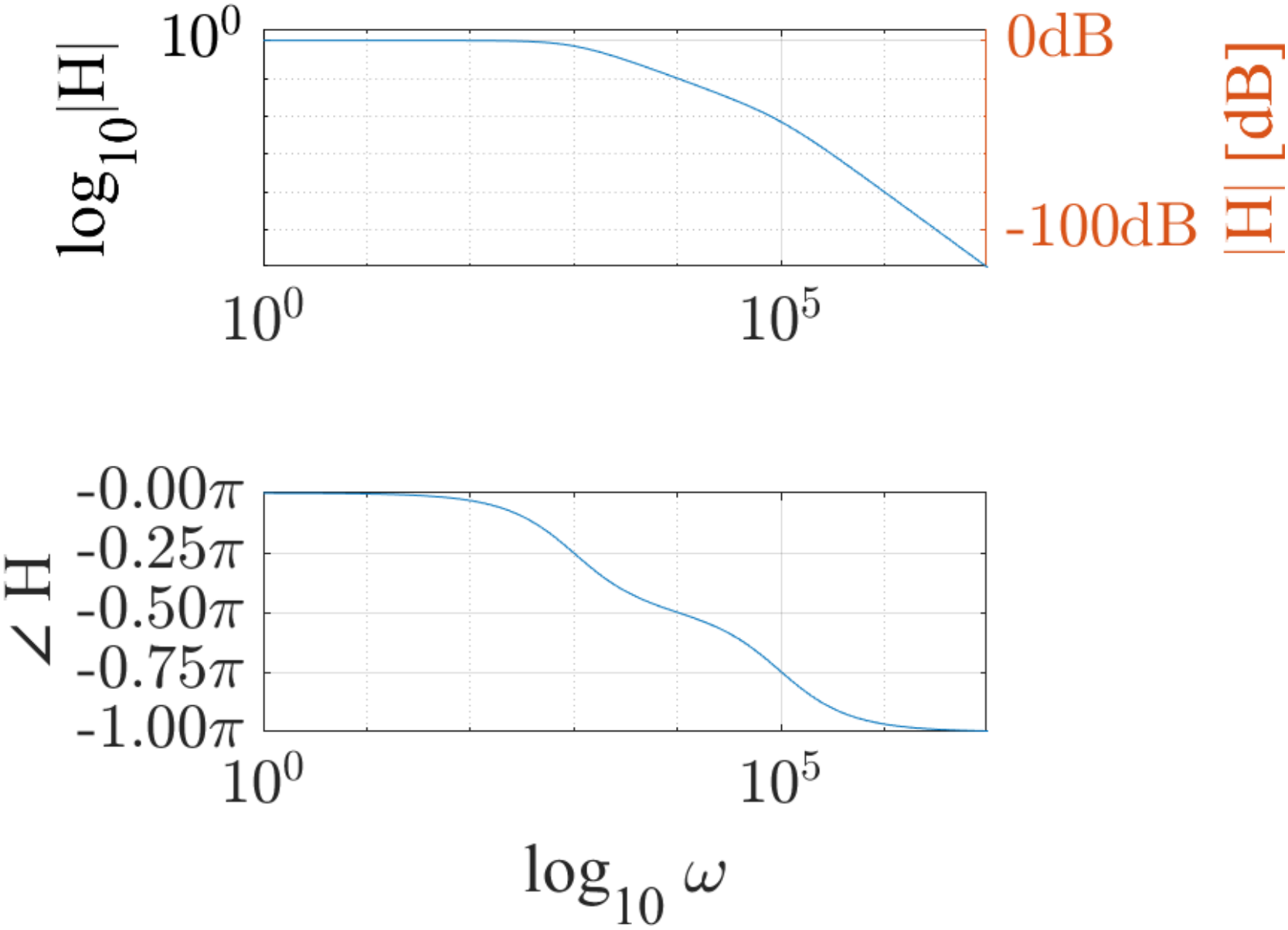
$$\frac{1}{Q^2} \leq 4 \Leftrightarrow |Q| \geq 0.5$$

The expression under the square root:

$$\sqrt{\frac{\omega_0^2}{Q^2} - 4\omega_0^2}$$

is positive and the two poles are negative real numbers (so the pole frequencies are positive). Our previous rules for sketching Bode plots apply.

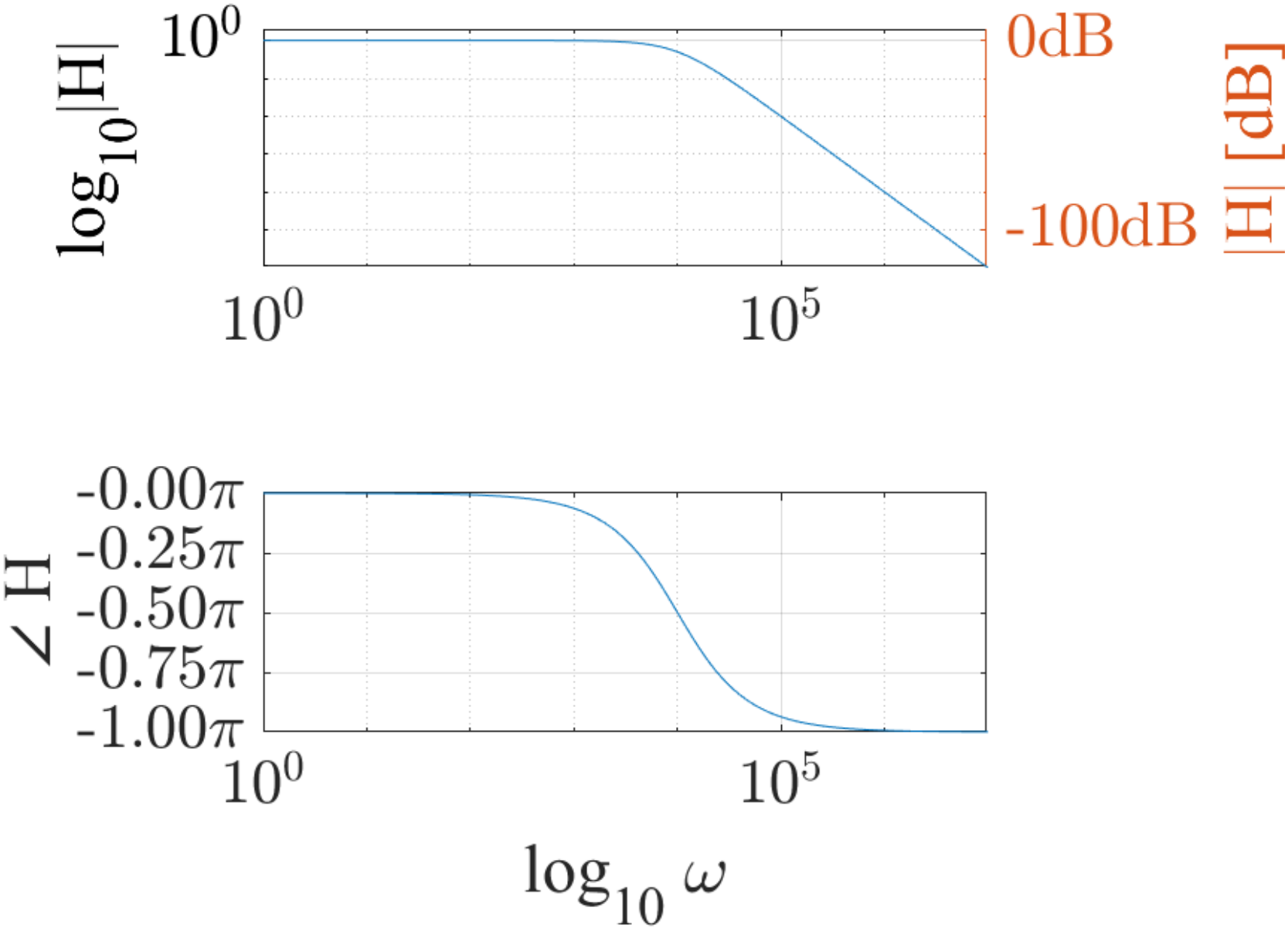
Q-factor ≤ 0.5 (2/2)



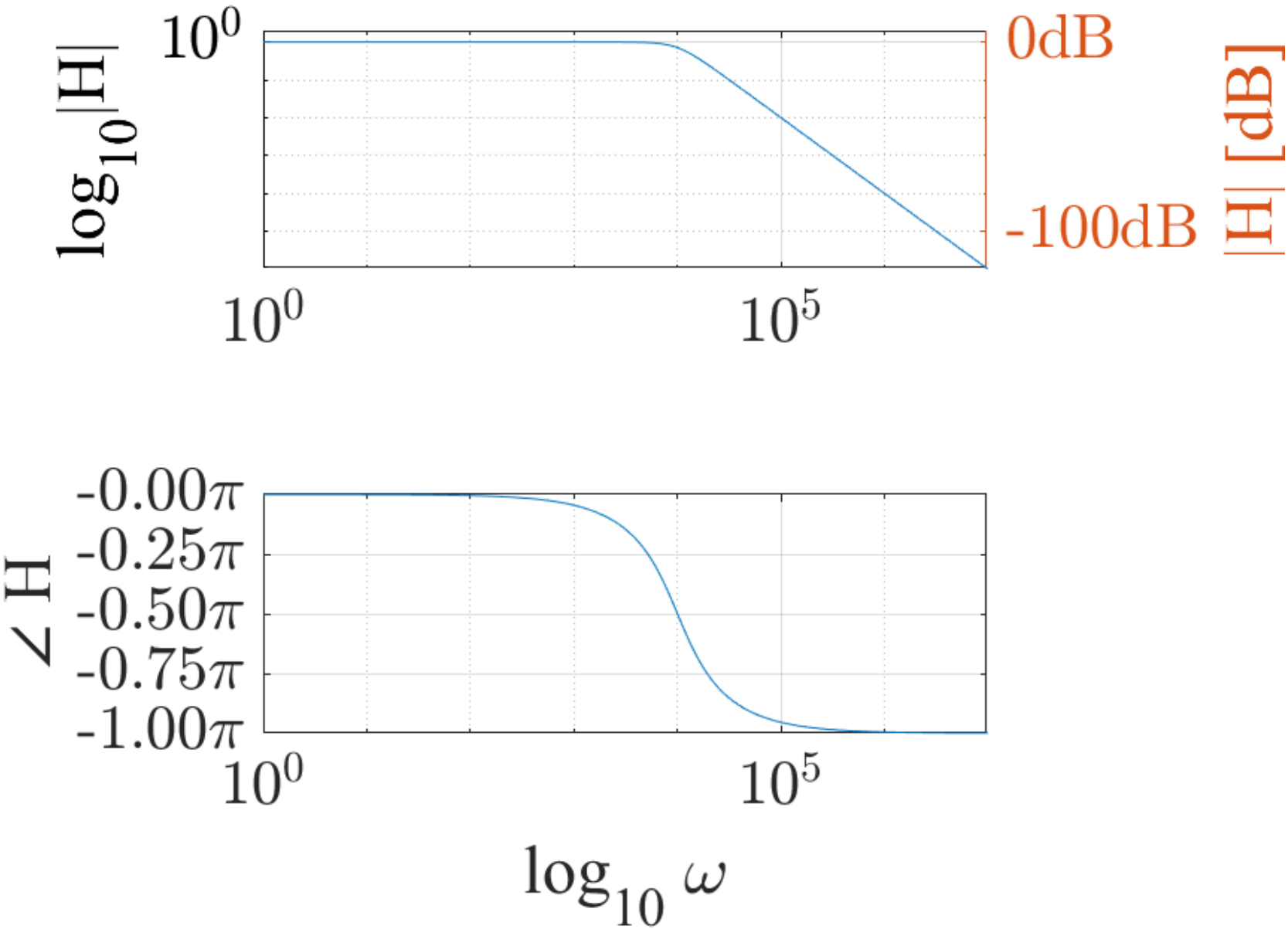
$\frac{1}{\sqrt{2}} \geq$ Q-factor > 0.5

Here the 'knee' in the transfer function becomes 'sharper'. At $Q = 0.5$ we get two identical real poles with pole frequency ω_0 and a -6dB at that frequency. Increasing Q some more the signal attenuation at ω_0 becomes less until it is at -3dB when $Q = \frac{1}{\sqrt{2}}$ (the maximally flat response), while the curve remains flat below that frequency and shows the same -40$\frac{\text{dB}}{\text{dec}}$ roll off beyond. The phase transition becomes more abrupt.

Q-factor = 0.5


10^0
$\log_{10}|H|$
0dB
-100dB
|H| [dB]
10^0
10^5
$\angle$ H
-0.00π
-0.25π
-0.50π
-0.75π
-1.00π
10^0
10^5
$\log_{10}\omega$

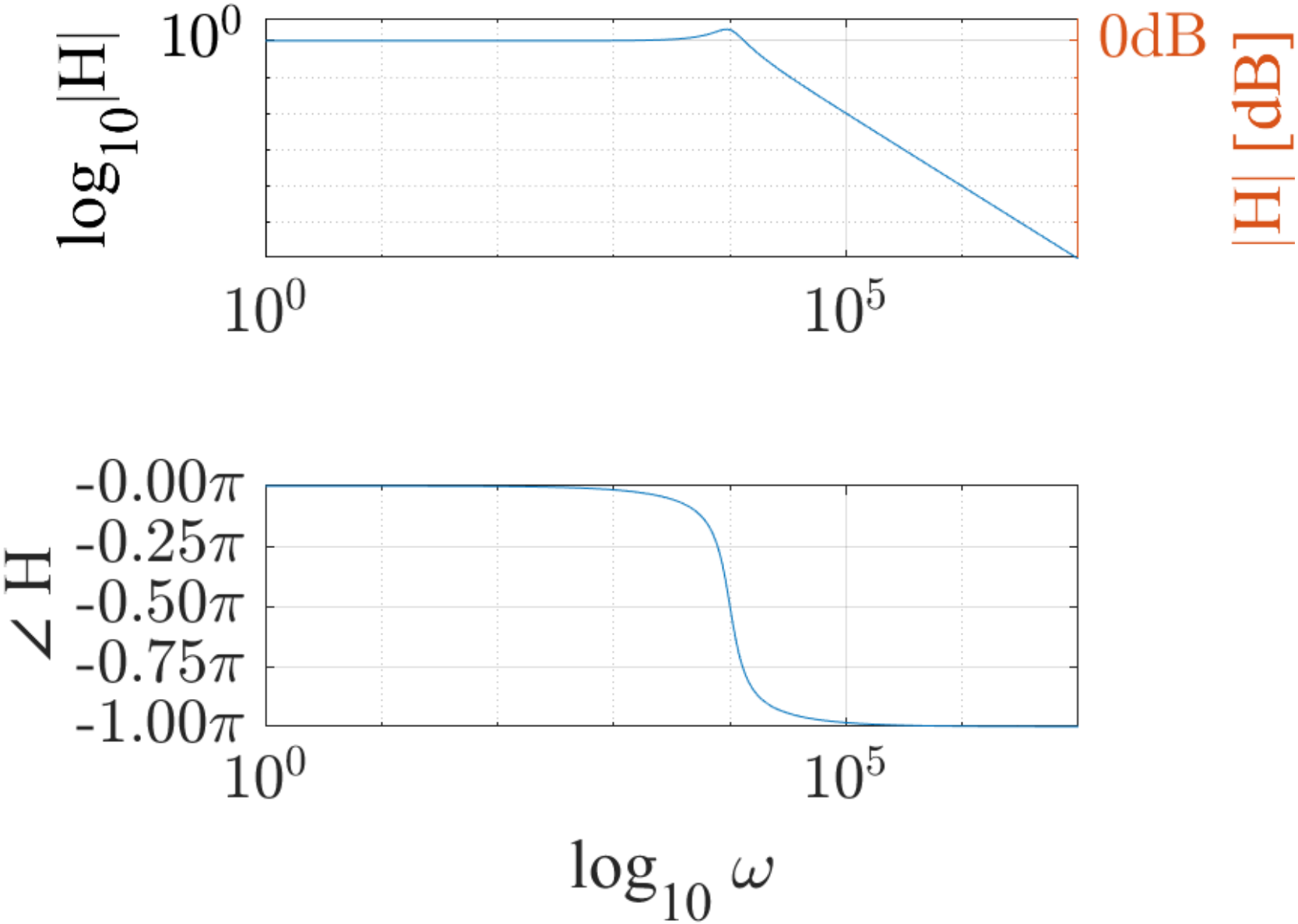
Q-factor $= \frac{1}{\sqrt{2}}$ (maximally flat response)



Q-factor $> \frac{1}{\sqrt{2}}$ (1/2)

Here finally we see clear resonance. The 'knee' develops a 'hump' and the gain exceeds the DC gain. The phase transition becomes even sharper.

Q-factor $> \frac{1}{\sqrt{2}}$ (2/2)



Condition for resonance

A mathematical proof is in the compendium appendix that shows that

$$\frac{1}{|A * B|} > 1 \Leftrightarrow \frac{2(r^2-1)}{(r^2+1)^2} > \mathrm{Im}(\frac{j\omega}{\omega_{p_{1,2}}})^2$$

Which only has a solution when $r > 1$

$$r := \frac{\left|\mathrm{Re}(\frac{j\omega}{\omega_{p_{1,2}}})\right|}{\mathrm{Im}(\frac{j\omega}{\omega_{p_{1,2}}})}$$

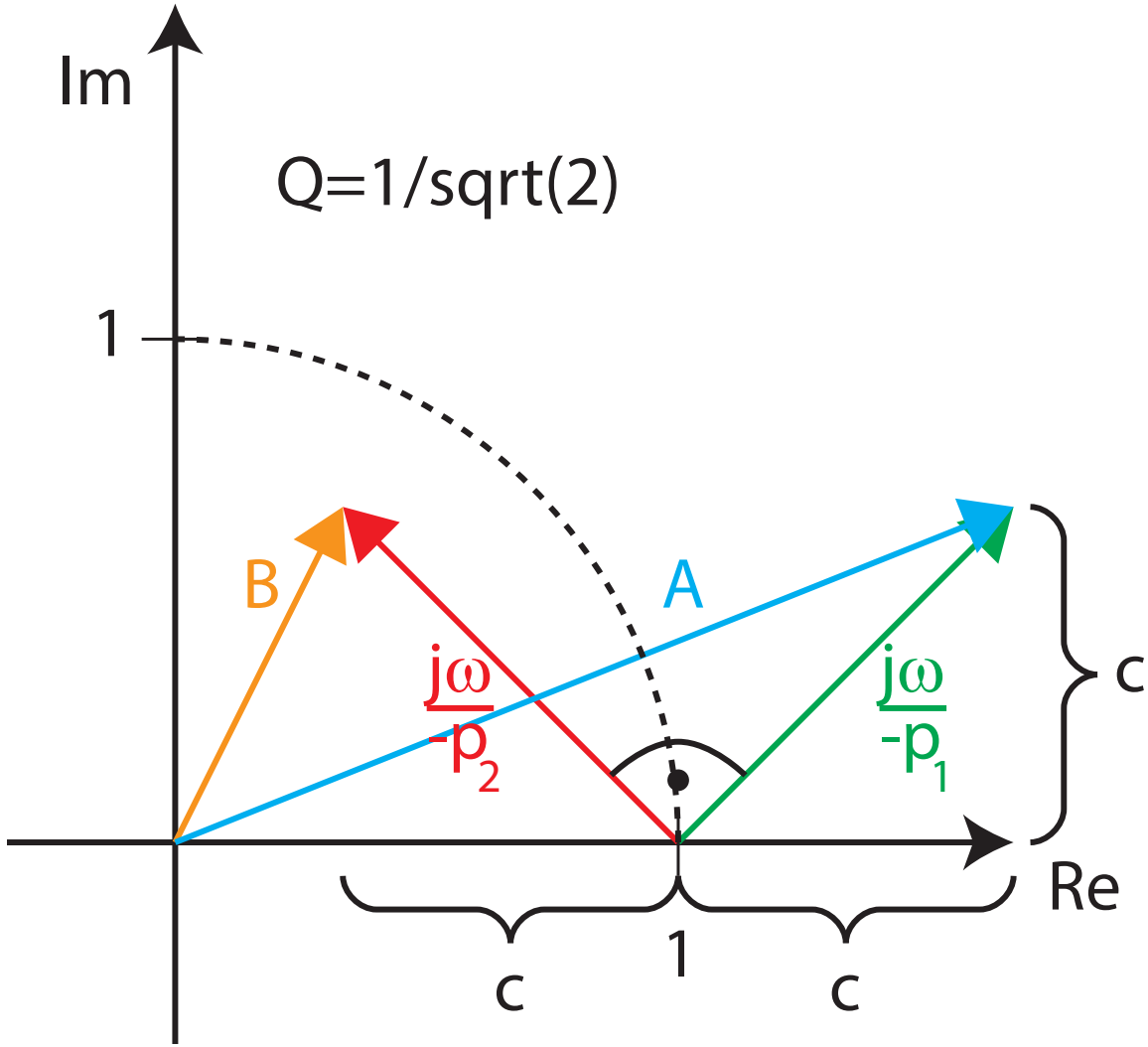
$$A := 1 + \frac{j\omega}{\omega_{p_1}}$$

$$B := 1 + \frac{j\omega}{\omega_{p_2}}$$

Thus:

$$Q > \frac{1}{\sqrt{2}} \Leftrightarrow r > 1$$

Graphical interpretation of what happens



Approximating higher order transfer functions with 2nd order

Remember the dominant pole approximation simply ignoring higher order terms? This can also be applied to 'cut off' higher order terms beyond the second order, if they are much smaller than the second order term.

Somewhat more precise than just simply 'cutting off' (especially if the lower frequency poles are NOT clearly dominant) is approximating higher frequency poles and zeros with a 'representative' ω_{eq}:

$$A\frac{(1+\frac{s}{\omega_{z_1}})...(1+\frac{s}{\omega_{zm}})}{(1+\frac{s}{\omega_{p_1}})...(1+\frac{s}{\omega_{pn}})} \approx A\frac{(1+\frac{s}{\omega_{z_1}})...(1+\frac{s}{\omega_{z_{y-1}}})}{(1+\frac{s}{\omega_{p_1}})...(1+\frac{s}{\omega_{p_{x-1}}})(1+\frac{s}{\omega_{\text{eq}}})}$$

Where:

$$\omega_{\text{eq}} \approx \frac{1}{\sqrt{\sum_{\omega_{p_i} \geq \omega_{p_x}} \frac{1}{\omega_{p_i}^2} - \sum_{\omega_{z_i} > \omega_{p_x}} \frac{2}{\omega_{z_i}^2}}}$$

Approximating higher order transfer functions with 2nd order

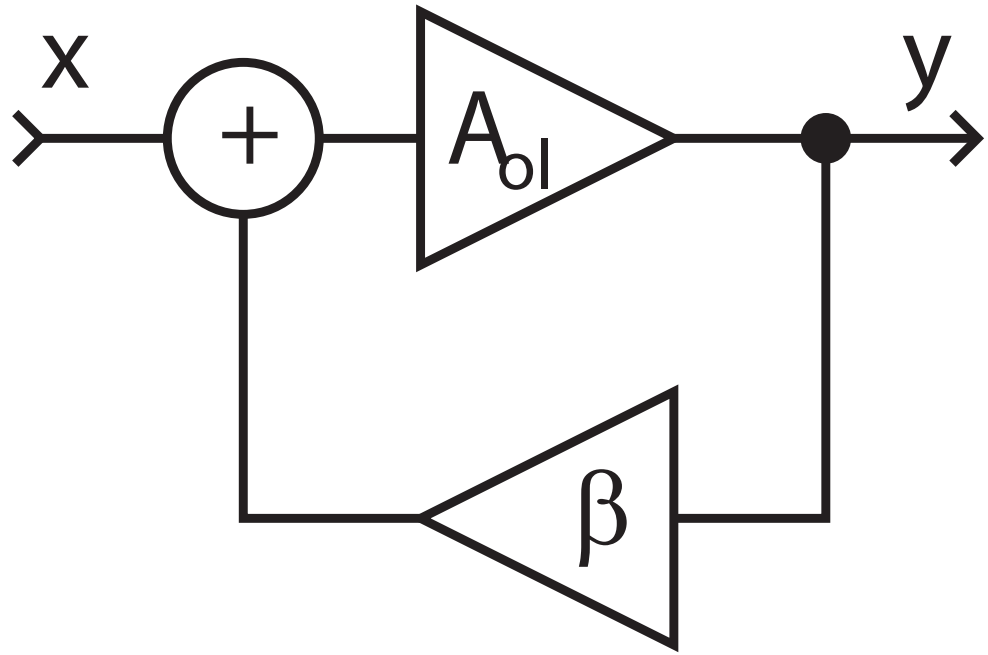
Why this equation for ω_{eq}?
Ignoring the lower frequency ($< \omega_{p_x}$) terms in the transfer function calling the remainder H we look for the cut off frequency ω_H $|H| = \frac{1}{\sqrt{2}} = -3\text{dB}$. Illustrating it for H being of second order with two poles and two zeros:

$$|H(\omega_H)|^2 = \frac{1}{2} = \frac{(1+\frac{\omega_H^2}{\omega_{z1}^2})(1+\frac{\omega_H^2}{\omega_{z2}^2})}{(1+\frac{\omega_H^2}{\omega_{p1}^2})(1+\frac{\omega_H^2}{\omega_{p2}^2})} \tag{3}$$

$$= \frac{1+\omega_H^2\left(\frac{1}{\omega_{z1}^2}+\frac{1}{\omega_{z2}^2}\right)+\omega_H^4\left(\frac{1}{\omega_{z1}^2\omega_{z2}^2}\right)}{1+\omega_H^2\left(\frac{1}{\omega_{p1}^2}+\frac{1}{\omega_{p2}^2}\right)+\omega_H^4\left(\frac{1}{\omega_{p1}^2\omega_{p2}^2}\right)} \tag{4}$$

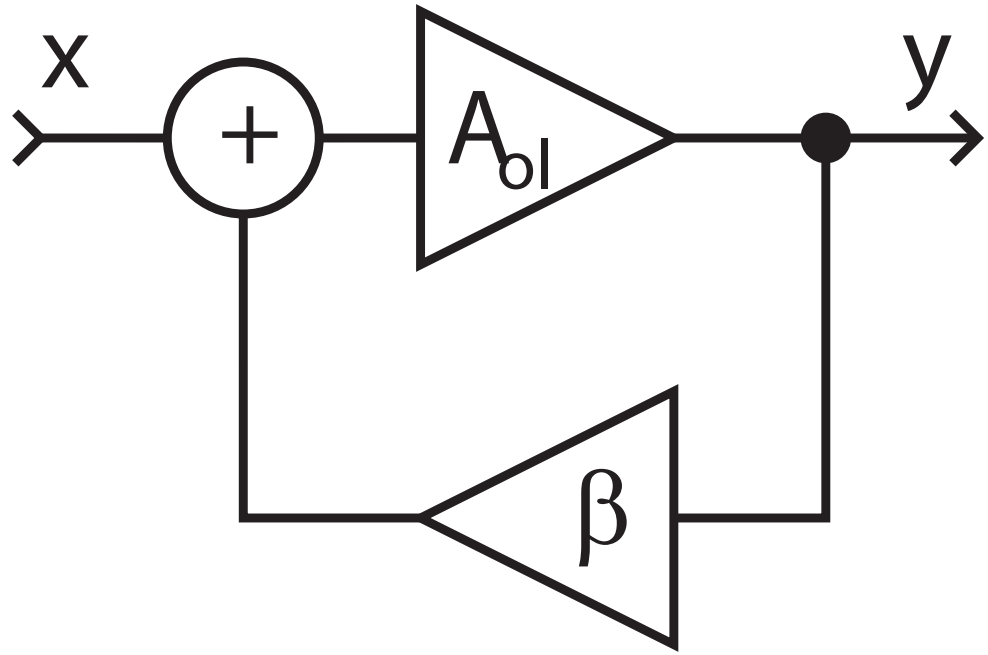
$$\Rightarrow \omega_H \approx \omega_{eq} = \frac{1}{\sqrt{\frac{1}{\omega_{p1}^2}+\frac{1}{\omega_{p2}^2}-\frac{2}{\omega_{z1}^2}-\frac{2}{\omega_{z2}^2}}} \tag{5}$$

General feedback loop concept



CAREFUL: Many other texts do subtract the feedback while I simply sum here to be more general, relying on the term $A_{ol}\beta$ being negative or smaller than 1 for stability. So in other books the sign in the equations containing $A_{ol}\beta$ may be inverted compared to here!

General feedback loop concept



$$A_{ol}\beta \geq 1 \quad \Rightarrow \quad A_{cl} = \infty$$

$$A_{ol}\beta < 1 \quad \Rightarrow \quad A_{cl} = \frac{A_{ol}}{1 - A_{ol}\beta} < \infty$$

$$A_{ol}\beta < 0 \quad \Rightarrow \quad A_{cl} < A_{ol}$$

$$A_{ol}\beta << -1 \quad \Rightarrow \quad A_{cl} \approx \frac{1}{\beta} \text{ (Often intentionally used this way.)}$$

Adding frequency dependency $A_{ol}(\omega)$

negative feedback means:

$$\angle A_{ol}(0)\beta = -\pi = -180^o$$

Adding phase shift in LP filters there is a critical frequency ω_{critical} where negative FB becomes real and positive:

$$\angle A_{ol}(\omega_{\text{critical}})\beta = -2\pi = -360^o$$

We are in serious trouble (sufficient condition) if at that point the absolute gain is bigger than 1 with infinite closed loop gain:

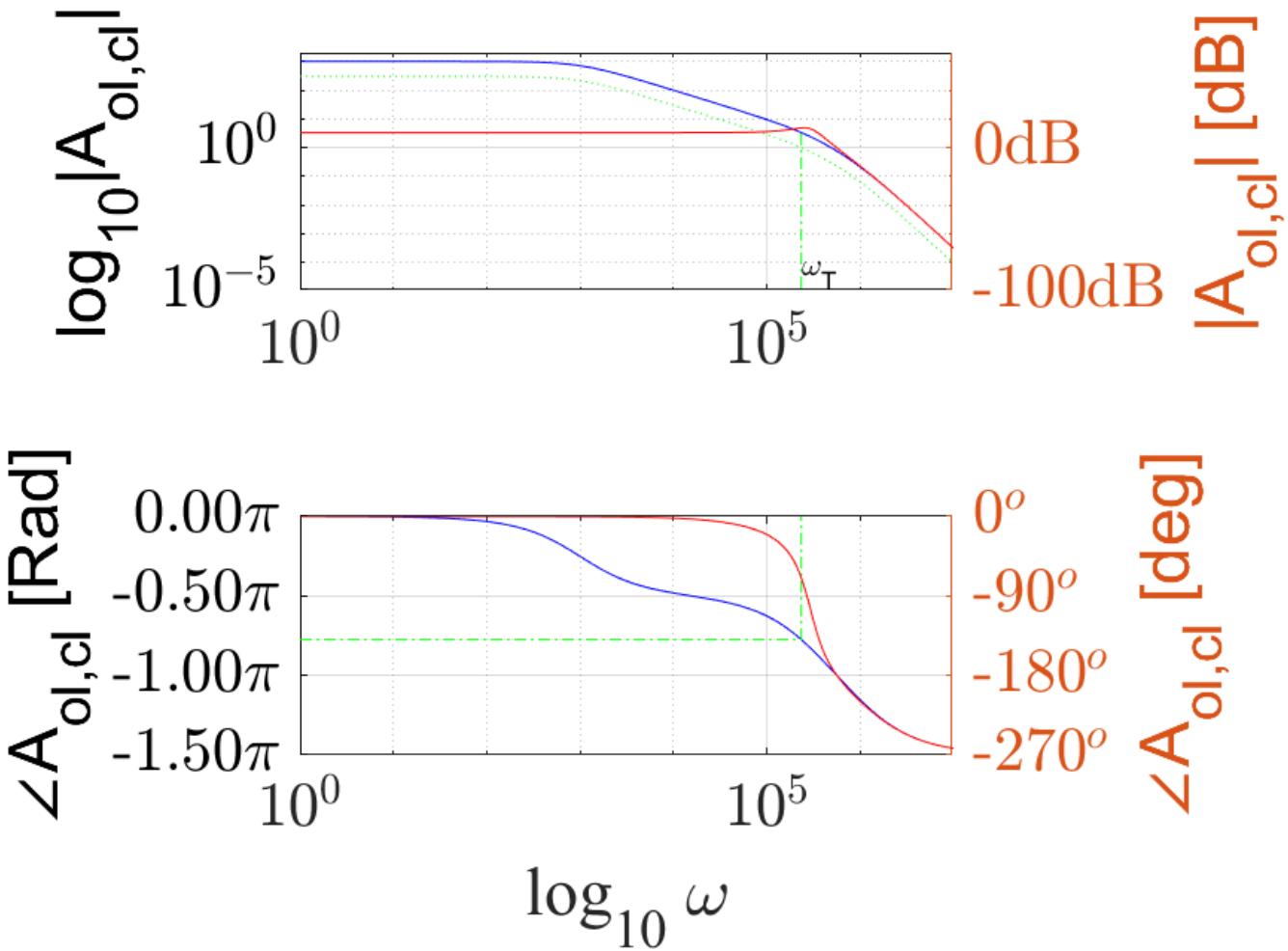
$$|A_{ol}(\omega_{\text{critical}})\beta| > 1 \Leftrightarrow A_{ol}(\omega_{\text{critical}})\beta > 1 \wedge A_{cl} = \infty$$

Phase margin (PM)

The phase margin is the angular distance of the phase at the unity gain frequency of the loop gain (!) from $-2\pi = -360^o$.

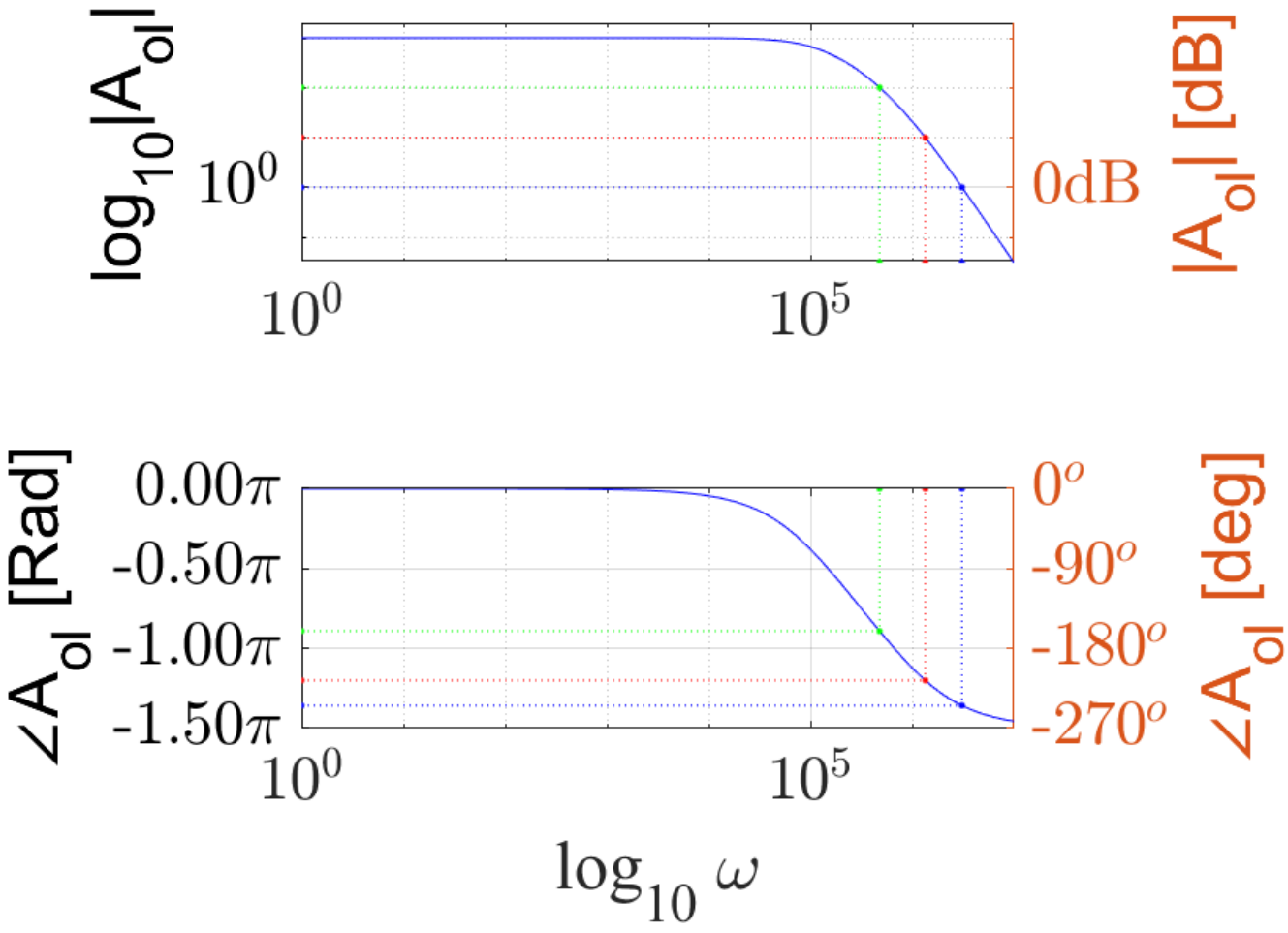
$$\text{PM} := \angle A_{ol}(\omega_T)\beta - \angle A_{ol}(\omega_{\text{critical}})\beta = \angle A_{ol}(\omega_T)\beta + 2\pi \qquad (6)$$

Example: PM=41°



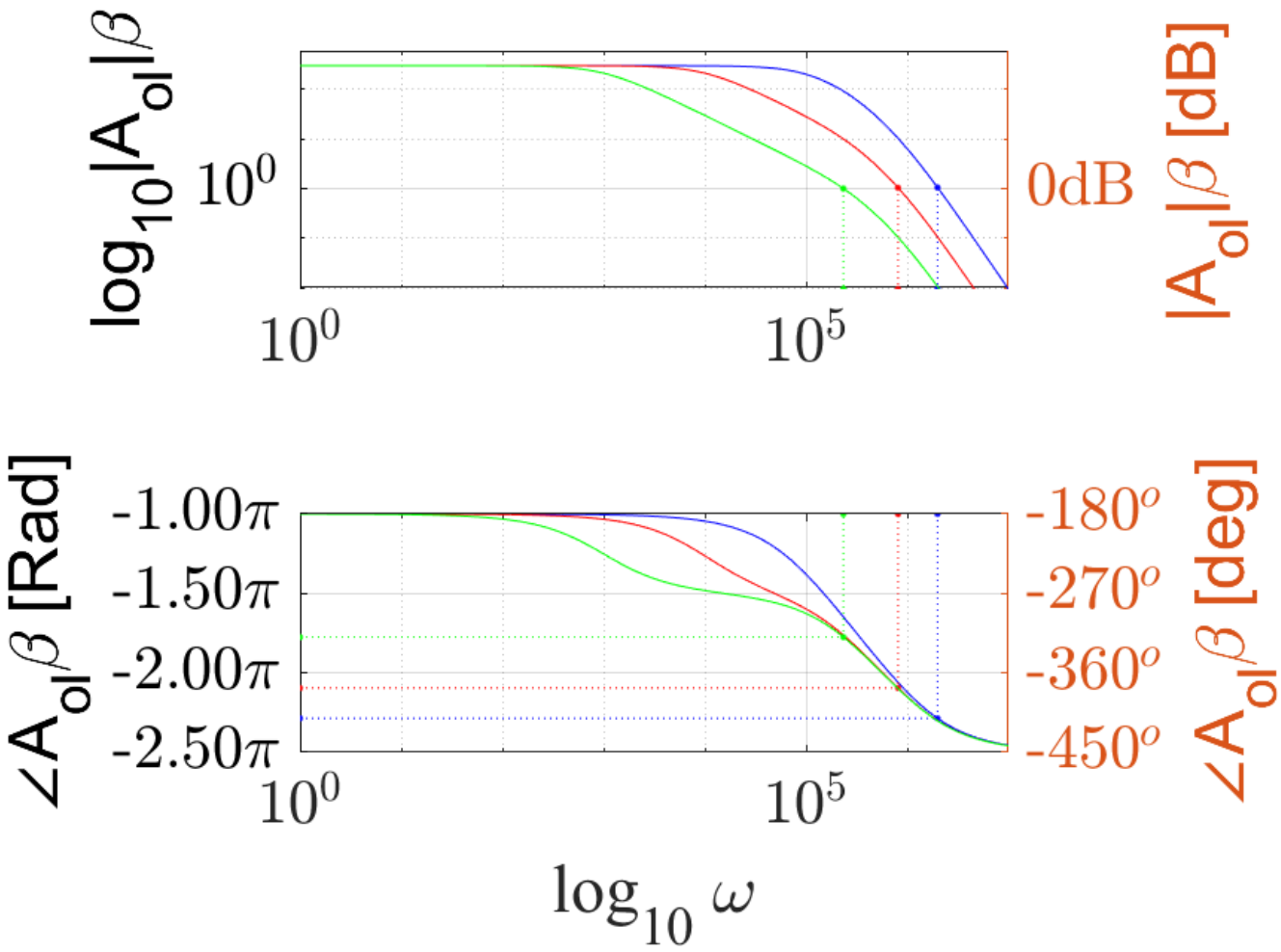
A third order open loop transfer function with no zeros where $A_{ol}(0) = 10^3$, $\beta = -\frac{3}{10}$, $\omega_{p_1} = 10^3$, $\omega_{p_2} = 10^{5.5}$, $\omega_{p_3} = 10^6$.
blue: open loop gain A_{ol}, red: closed loop gain A_{cl}, green dotted: loop gain $A_{ol}\beta$

Example: influence of β



Where $\beta = -1$ (blue, worst case), $\beta = -\frac{1}{10}$ (red), $\beta = -\frac{1}{100}$ (green)

Example: influence of 'separating' the first pole



Where $\omega_{p_1} = 10^5$Rad (blue, worst case), $\omega_{p_1} = 10^4$Rad (red), $\omega_{p_1} = 10^3$Rad (green), and the next $\omega_{p_2} = 10^{5.5}$ and $A_{ol}(0)\beta = -300$

Finding the 'loop'?

Not easy in the general case! The theory is appropriate for a unidirectional signal at the point where one would need to define the 'break' of the loop. So the method is best suited where the system is intentionally designed as a loop.

Relation between open loop poles and closed loop poles and Q-factor

If A_{ol} is a known second order transfer function we can compute the poles of the closed loop transfer function A_{cl}, as well as its Q-factor Q_{cl} and characteristic frequency ω_{0cl}.
For the poles we solve:

$$0 = 1 + A_{ol}\beta$$

getting:

$$p_{cl1}, p_{cl2} = \frac{-(\omega_{p_1} + \omega_{p_2}) \pm \sqrt{(\omega_{p_1} + \omega_{p_2})^2 - 4\omega_{p_1}\omega_{p_2}(1 - A_{ol}(0)\beta)}}{2}$$

$$\Rightarrow$$

$$Q_{cl} = \frac{\sqrt{\omega_{p_1}\omega_{p_2}(1 - A_{ol}(0)\beta)}}{\omega_{p_1} + \omega_{p_2}}$$

$$\omega_{0cl} = \sqrt{\omega_{p_1}\omega_{p_2}(1 - A_{ol}(0)\beta)}$$

Relation between PM and Q-factor

In case of a 'real poled' second order A_{ol} we can also get the PM from the ratio of the open (!) loop second pole frequency ω_{p_2} and the loop gain's (!) unity gain frequency ω_T, here presented as a table:

PM	35.09	40.65	50.35	54.18	60.75	64.02	70.23	75.54	79.55	85.00
$\frac{\omega_T}{\omega_{p_{ol2}}}$	1.48	1.20	0.85	0.74	0.58	0.50	0.37	0.27	0.20	0.10
Q	1.60	1.35	1.05	0.97	0.82	0.75	0.63	0.53	0.45	0.32

NOTE: if one merely *approximates* a higher order A_{ol} with a second order function using ω_{eq}, the PM in this table is overestimated. So use this table with confidence only if A_{ol} really is of 2nd order.