Ex03 — Frequency-domain representation of signals

Delivery deadline: **16 September** 2022.

We highly recommend you to collaborate and interact with each other when working on these tasks. Still, the exercises shall be solved independently and must be assessed in the weekly submission sessions. It is expected that everyone is familiar with the approval requirements for exercises. This is described on the course homepage: `http://www.uio.no/studier/emner/matnat/ifi/IN3190/h22/curriculum-related/`

Observe: The sum of points in total is 13, while the maximum score obtainable in one week is 10.

Goals:

- The Discrete-Time Fourier Transform (DTFT) is a key transform in the analysis and processing of discrete signals and systems that are aperiodic – as are most signals that are recorded by real-world sensors. Here, you will get some experience in calculating the DTFT analytically for a set of basic discrete signals. The DTFT transforms our signal from a discrete (sampled) domain to a continuous and periodic frequency-domain where we typically use the symbol ω for the continuous angular frequency.
- When the discrete signal is periodic with a fundamental period N, the Discrete-Time Fourier Series (DTFS) is a more relevant transform. This will not be applied as often as DTFT in the coure. But still, we would like you to have some hands-on experience with it and you are therefore offered to work on one such exercise this week. In this task, you will also practice some calculation using complex exponentials – thereby building some additiona intuition on how these can interplay to form real-valued trigonomtric funcions.
- Parseval's theorem for a power signal says that the power of a signal is the same in the time as in the Fourier domain. You will verify this in one of the problems.
- You will also be tasked to apply the convolution operator to a real-world signal, and interpret your result.

Exercise 1 — DTFT task from Chapter 2 in Rao & Swamy 4 Points

Calculate the DTFT for the following signals. $u(n)$ signifies the step function, which is zero for negative n and 1 otherwise.

(a) $x_1(n) = u(n) - u(n-5)$

(b) $x_2(n) = \alpha^n(u(n) - u(n-8)), \quad |\alpha| < 1$
Hint: Re-write in terms of two geometric series, which have closed-form sum expressions

(c) $x_3(n) = n(\frac{1}{2})^{|n|}$ *Hint:* Split the expression into non-negative and negative n

(d) $x_4(n) = \alpha^{|n|} \sin(\omega_0 n)$

Exercise 2 — Discrete-time Fourier Series (DTFS) 4 Points

Let's consider a signal which repeats itself with a fundamental period $N = 6$. Hence if you offset the signal by 6 samples in either direction, it will still look the same:

$$x(n) = \{\ldots, 1, 0, 1, 2, \underset{\uparrow}{3}, 2, 1, 0, 1, 2, 3, 2, 1, 0, 1, 2, 3, 2, 1, 0, 1, 2, 3, 2, \ldots\}$$

a) Sketch the signal $x(n)$ and calculate its discrete-time Fourier Series (DTFS). Use the following DTFS definition:

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}.$$

Hints:

It might be useful to start out from the DTFS definition and then insert the fundamental period $N = 6$. Thereafter, the following analysis can be made for any k:

$$\begin{aligned}
c_k &= \frac{1}{6}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}k\cdot n} = \frac{1}{6}\sum_{n=0}^{6-1} x(n)e^{-j\frac{2\pi}{6}k\cdot n} = \frac{1}{6}\sum_{n=0}^{5} x(n)e^{-j\frac{\pi}{3}k\cdot n} \\
&= \frac{1}{6}(3e^{-j\frac{\pi}{3}k\cdot 0} + 2e^{-j\frac{\pi}{3}k\cdot 1} + 1e^{-j\frac{\pi}{3}k\cdot 2} + 0e^{-j\frac{\pi}{3}k\cdot 3} + 1e^{-j\frac{\pi}{3}k\cdot 4} + 2e^{-j\frac{\pi}{3}k\cdot 5}) \\
&= \frac{1}{6}(3 + 2e^{-j\frac{\pi}{3}k} + e^{-j\frac{2\pi}{3}k} + e^{-j\frac{4\pi}{3}k} + 2e^{-j\frac{5\pi}{3}k}) \\
&= \frac{1}{6}(3 + 2e^{-j\frac{\pi}{3}k} + e^{-j\frac{2\pi}{3}k} + e^{j\frac{2\pi}{3}k} + 2e^{j\frac{\pi}{3}k}) \\
&= \frac{1}{6}\big(3 + e^{j\frac{2\pi}{3}k} + e^{-j\frac{2\pi}{3}k} + 2\left(e^{j\frac{\pi}{3}k} + e^{-j\frac{\pi}{3}k}\right)\big)
\end{aligned}$$

In the last step above, you almost a nice expression ready. Massage this expression a little bit more by applying the relation $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$. Then you see that the expression is real-valued and has three terms: the integer number 3 and two different cosine terms.

Finally, calculate c_k for all $k = 0, 1, 2, 3, 4, 5$, and you have the answer which is a discrete sequence of 6 numbers.

b) Verify Parseval's theorem from the results in **a)**, by calculation the signal effect over a period in the time-domain and compare this to what you get in the frequency domain.

a) $\boxed{c_k = \{\frac{3}{2}, \frac{2}{3}, 0, \frac{1}{6}, 0, \frac{2}{3}\}}$ b) $\boxed{\frac{19}{6}}$

Exercise 3 — Based on 4.32 of Manolakis & Ingle: 2 Points

Given that the signal $x(n)$ has its known DTFT $X(e^{j\omega})$, find the DTFT of the following signals. Express your results in terms of $X(e^{j\omega})$ and tell what DTFT properties you have applied:

a) $x_1(n) = 2x(n+2) + 3x(3-n)$ *Hint:* apply the time-shift and folding properties

c) $x_3(n) = 2e^{j0.5\pi(n-2)}x(n+2)$,

d) $x_4(n) = (x(n) - x^*(-n))/2$,

e) $x_5(n) = j^n x(n+1] + j^{-n}x(n-1)$.

Exercise 4 — A simple filter applied to a real-world signal: 3 Points

You are provided with a real recording from a highly sensitive instrument that measured propagating waves from the massive 2022 Hunga Tonga–Hunga Ha'apai eruption and tsunami.

- For more information about the event, see the following Wikipedia link and the references therein.
- You can download the signal $x(n)$ in HDF format from the following link:
  `https://www.uio.no/studier/emner/matnat/ifi/IN3190/h22/weekly_tasks/am.ra461_unfilt.h5`
- You can download a filter sequence from the following link:
  `https://www.uio.no/studier/emner/matnat/ifi/IN3190/h22/weekly_tasks/window.mat`
- This signal was recorded with 100 samples per second. The sensor is located at latitude: -6.864864865, longitude: 107.5832468 and elevation: 826 meter, on the island of Java, Indonesia.

You are asked to perform a basic filtering of this signal in order to facilitate interpretation.

a) Read the data. In Matlab, you can use the following call:
`data = h5read('AM.RA461_unfilt.h5','/waveforms/000_AM.RA461.00.HDF_2022-01-15T04:00:00_72000.0s');`,
while in Python you can achieve the same using the `h5py` tools or, for example, `xarray.open_dataset`
More info about `h5py`: `https://docs.h5py.org/en/stable/`.

b) Plot the recorded signal.

c) Read the filter sequence $h(n)$. You can use `load` in Matlab, or `scipy.io.loadmat` in Python to load it.

d) Plot the filter sequence.

e) Apply a numerical implementation of convolution to filter the Hunga Tonga signal. Hence, you shall calculate the convolution $y(n) = x(n) * h(n)$. You can, for example, use the convolution function you implemented in the exercise last week. However, if that implementation turns out to be too slow: please feel free to use an optimized implementation like `conv` in Matlab or `numpy.convolve`.

f) Plot $y(n) = x(n) * h(n)$

g) Look at $y(n)$ in the context of $x(n)$. Does it appear like the filtering was low-pass (keeping slowly-varying aspects of the signal), or high-pass (keeping rapidly-varying aspects of the signal)?

h) The recording starts on 15 January 2022 at 4h UTC. Look at the filtered signal $y(n)$ and estimate approximately the time when the sound wave arrived at the sensor.