Ex8 — DFT

Delivery deadline: **4 November** 2022.

We highly recommend you to collaborate and interact with each other when working on these tasks. Still, the exercises shall be solved independently and must be assessed in the weekly submission sessions. It is expected that everyone is familiar with the approval requirements for exercises. This is described on the course homepage: `http://www.uio.no/studier/emner/matnat/ifi/IN3190/h22/curriculum-related/`

Observe:
The sum of points in total for the tasks distributed the current week is 12, while the maximum score obtainable in one week is 10. The score registered will hence be obtained using the formula `min(your_score_sum, 10)`.

Goals and reflections:

- The DFT is the Fourier transform variant that is the most commonly used in practical numerical analysis of data. (Although it's a specific algorithm implementation called FFT that is in most often applied in practice. Still, the output is the same as the output using the direct definition of the DFT.)
- Fully comprehending the effects that sampling in the time domain has on the Fourier-domain representation of a signal is of great value for achieving an intuitive understanding of the core of digital signal processing.

Exercise 1 — DFT from its definition — Points: 4

Compute the DFT from its definition, for the following signals:

a) $x(n) = \{1, 2, 1, 2\}$ b) $x(n) = \{2, 1, 3, 0, 4\}$
c) $x(n) = \{2, 2, 2, 2\}$ d) $x(n) = \{1, 0, 0, 0, 0, 0, 0, 0\}$

- Hint: $e^{-j\pi/2} = -j, \quad e^{-j\pi} = -1, \quad e^{-j\pi/2} = -j, \quad e^{-j3\pi/2} = j$
- Hint: The DFT exhibits conjugate symmetry around $k = N/2$. Hence, $X_{\text{DFT}}(N-k) = X^{\star}_{\text{DFT}}(k)$. This way, one can save some calculation effort if only calculating the value for indices $k \leq N/2$, and then use the conjugate symmetry relation for the rest of the indices.

a) $\boxed{X_{\text{DFT}}(k) = \{6, 0, -2, 0\}}$ b) $\boxed{X_{\text{DFT}}(k) = \{10, 1.12 + j1.09, -1.12 + j4.62, -1.12 - j4.62, 1.12 - j1.09\}}$
c) $\boxed{X_{\text{DFT}}(k) = \{8, 0, 0, 0\}}$ d) $\boxed{X_{\text{DFT}}(k) = \{1, 1, 1, 1, 1, 1, 1, 1\}}$

Exercise 2— Sampling and aliasing — Points: 8

The periodic signal $x_a(t) = 5\cos(200\pi t + \pi/6) + 4\sin(300\pi t) = \frac{5}{2}\left(e^{j200\pi t}e^{j\pi/6} + e^{-j200\pi t}e^{-j\pi/6}\right) + \frac{4}{j2}\left(e^{j300\pi t} - e^{-j300\pi t}\right)$ is sampled at a rate of $F_T = 1$ kHz to obtain the discrete-time signal $x(n)$. t is measured in seconds.

(a) Determine the spectrum $X(e^{j\omega})$ of $x(n)$.

Plot its magnitude as a function of normalized angular frequency ω in $\frac{\text{rad}}{\text{sample}}$ and as a function of frequency F in Hz. Plot the spectrum for $-2.5 \leq \omega/\pi \leq 2.5$ and $-2F_T \leq F \leq 2F_T$.

Explain whether the original signal $x_a(t)$ can be recovered from $x(n)$.

Hints:

- First find $X_a(j\Omega)$ using the CTFS (the standard continuous-time Fourier transform). Thereafter, calculate $X(e^{j\omega})$, which is a scaled and periodized version of $X_a(j\Omega)$. Remember also that $\omega = \Omega T = \Omega/F_T = 2\pi F/F_T$. Likewise, we have that $\Omega_T = 2\pi F_T$.
- If you use a decomposition of $x_a(t)$ of the form

$$x_a(t) = A\left(e^{j\Omega_1 t}e^{j\phi} + e^{-j\Omega_1 t}e^{-j\phi}\right) + B\left(e^{j\Omega_2 t} - e^{-j\Omega_2 t}\right), \tag{1}$$

  then you in fact already have found $X_a(j\Omega)$ "by inspection." Because we have that the inverse CTFS of a periodic signal at et periodisk signal is a decomposition into exponents on this form: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t}$.

  Therefore, for $x_a(t)$ given on the form (1), this will in the Fourier domain correspond to

$$X_a(j\Omega)\begin{cases} Ae^{j\phi}, & \text{for } \Omega = \Omega_1 \\ Ae^{-j\phi}, & \text{for } \Omega = -\Omega_1 \\ B, & \text{for } \Omega = \Omega_2 \\ -B, & \text{for } \Omega = -\Omega_2 \\ 0 & \text{otherwise} \end{cases}$$

- Remember that sampling in time yields a periodization (generating infinitely many scaled copies) in the Fourier domain, with a periodicity Ω_T.
- Rembember that $e^{jy} = \cos(y) + j\sin(y)$ which gives $\cos(y) = \dfrac{e^{jy} + e^{-jy}}{2}$ and $\sin(y) = \dfrac{e^{jy} - e^{-jy}}{2j}$.
- Det kan også være nyttig å bruke at $e^{j(y+\phi)} = e^{jy}e^{j\phi}$.

(b) Repeat part (a) for $F_T = 500$ Hz.

(c) Repeat part (a) for $F_T = 100$ Hz.

(d) Comment on your results: For what sampling frequencies can the original continuous signal be reconstructed from the sampled signal? What happens when the sampling frequency is too low?