



Chapter 2

Logics

Course “Specification and Verification of Parallel Systems”

Martin Steffen

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Chapter 2

Learning Targets of Chapter “Logics”.

The chapter gives some basic information about “standard” logics, namely propositional logics and (classical) first-order logics



Chapter 2

Outline of Chapter “Logics”.

Introduction

Propositional logic

First-order logic

Modal logics



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What's logic?

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General aspects of logics

- truth vs. provability
 - when does a formula *hold*, is *true*, is *satisfied*
 - valid
 - satisfiable
- syntax vs. semantics/models
- model theory vs. proof theory
- connection to computation, calculation, programs



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Many different logics

- propositional and first-order logics (classical or otherwise)
- higher-order logics
- modal and temporal logics
- “program” logics
- special purpose logics, domain-specific constraints . . .
- fuzzy, probabilistic



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Propositional logic

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Syntax (formulas, aka propositions)



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$\varphi ::=$ propositions
 $p \mid \top \mid \perp$ atomic propositions
 $\mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg \varphi \mid \varphi \rightarrow \varphi$ compound propositions

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Semantics: the meaning (here of propositions)

different truth values (we only consider $\mathbb{B} = \{\top, \perp\}$) Variable assignment:

$$\sigma : AP \rightarrow \mathbb{B} .$$

“Semantic brackets”:

$$\llbracket _ \rrbracket^- : (AP \rightarrow \mathbb{B}) \rightarrow \Phi \rightarrow \mathbb{B} .$$

Model relation or satisfaction relation:

$$\sigma \models \varphi \text{ iff } \llbracket \varphi \rrbracket^\sigma = \top$$

How are $\sigma \models \varphi$ and $\llbracket \varphi \rrbracket^\sigma$ defined?



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Definition of $\sigma \models \varphi$ and $\llbracket \varphi \rrbracket^\sigma$

remember: $\sigma \models \varphi$ iff $\llbracket \varphi \rrbracket^\sigma = \top$ and $\sigma \not\models \varphi$ iff $\llbracket \varphi \rrbracket^\sigma = \perp$

- $\sigma \models p$ iff $\sigma(p) = \top$ for $p \in AP$
- $\sigma \models \top$
- $\sigma \not\models \perp$
- $\sigma \models \neg \varphi$ iff $\sigma \not\models \varphi$
- $\sigma \models \varphi \wedge \varphi'$ iff $\sigma \models \varphi$ and $\sigma \models \varphi'$
- $\sigma \models \varphi \vee \varphi'$ iff $\sigma \models \varphi$ or $\sigma \models \varphi'$
- $\sigma \models \varphi \rightarrow \varphi'$ iff $\sigma \not\models \varphi$ or $\sigma \models \varphi'$



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Interesting problems



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model checking

$$\sigma \stackrel{?}{\models} \varphi$$

(in P)

satisfiability

$$\stackrel{?}{\models} \varphi$$

(NP-complete)

validity

$$\stackrel{?}{\models} \varphi$$

(co-NP-complete)

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Proof theory

- different questions about formula: how to *infer*/derive/deduce formulas (from others), i.e., how to do a proof?
- mechanical process
- *proof* = deduction (sequence or tree of steps)
- theorem = formula that is derivable
 - corresponds to a formula which is valid (semantical)
- soundness and completeness of interest

How could such a process look like?¹

¹Actually, for propositional logic the problem is so simple, the earlier definition of $\sigma \models \varphi$ can be seen as a recursive procedure to calculate satisfaction for propositional logic.



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Deductions and proof systems



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A **proof system** for a given logic consists of

- **axioms** (or *axiom schemata*), which are formulae assumed to be true, and
- **inference rules**, of approx. the form

$$\frac{\varphi_1 \quad \dots \quad \varphi_n}{\psi}$$

- $\varphi_1, \dots, \varphi_n$ are **premises** and ψ **conclusion**.

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A simple form of derivation

Derivation of φ

Sequence of formulas, where each formula is

- an axiom or
- can be obtained by applying an inference rule to formulae earlier in the sequence.

- Let Γ be a set of assumptions/hypotheses:

$$\Gamma \vdash \varphi$$

- special case (theorems): $\vdash \varphi$
- proof = derivation



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A proof system for propositional logic

$$\frac{}{\varphi \rightarrow (\psi \rightarrow \varphi)} \text{Ax}_1$$

$$\frac{}{(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))} \text{Ax}_2$$

$$\frac{}{((\varphi \rightarrow \perp) \rightarrow \perp) \rightarrow \varphi} \text{DN}$$

$$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi} \text{MP}$$

Task: prove $p \rightarrow p$



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A small derivation



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Example

$p \rightarrow p$ is a theorem of the proof system:

$$\begin{array}{ll} (p \rightarrow ((p \rightarrow p) \rightarrow p)) \rightarrow & \text{Ax}_2 (\varphi = p, \psi = (p \rightarrow p), \chi = p) \\ ((p \rightarrow (p \rightarrow p)) \rightarrow (p \rightarrow p)) & (1) \end{array}$$

$$p \rightarrow ((p \rightarrow p) \rightarrow p) \quad \text{Ax}_1 (\varphi = p, \psi = (p \rightarrow p)) \quad (2)$$

$$(p \rightarrow (p \rightarrow p)) \rightarrow (p \rightarrow p) \quad \text{MP on (1) and (2)} \quad (3)$$

$$p \rightarrow (p \rightarrow p) \quad \text{Ax}_1 (\varphi = p, \psi = p) \quad (4)$$

$$p \rightarrow p \quad \text{MP on (3) and (4)} \quad (5)$$

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Proof systems and proofs: remarks



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- “definitions” from the previous slides: not very formal
- many different approaches of how to draw conclusions exist, the one sketched on the previous slide
 - works with “sequences”
 - corresponds to the historically oldest approach to proof systems (“Hilbert-style”)
 - **but** usually sound & complete

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Summary: proof theory



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- decidable (for propositional logics), so it's a “trivial” problem in that sense
- truth tables (brute force)
- deduction systems
- one can try to do better, different derivation strategies (resolution, refutation, ...)

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Logics so far: rather sterile

- propositional logic: cannot speak about “things” just regulates “truth” in a logical vacuum, but not “truth **of** something”
- **predicates**

Example (“even” as predicate)

- $even(4)$: true,
- $even(5)$, $even(4 + 1)$: false,
- $even(x)$: depends on value of *variable* x



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- **Sort**
 - name of a domain (like Nat)
 - restricted form of type
- single-sorted vs. multi-sorted case
- single-sorted
 - one sort only
 - “degenerated”
 - *arity* = number of arguments (also for relations)

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- fixes the “syntactic playground”
- selection of symbols
 - *functional*
 - *relational*

together with arity (or sort-information)

Definition (Signature (single-sorted))

A *signature* $\Sigma = (\Sigma_f, \Sigma_{rel}, ar)$ consists of two finite, disjoint sets of functional and relational symbols together with a function $ar : \Sigma_f + \Sigma_{rel} \rightarrow \mathbb{N}$, fixing the *arity* for each symbol.



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- given: signature Σ
- set of variables X (with typical elements $x, y', \dots$)

Definition (Σ -terms)

The *set of terms* w.r.t. a signature Σ and a set of variables X is recursively defined by the following grammar:



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- given: signature Σ
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Definition (Σ -terms)

The *set of terms* w.r.t. a signature Σ and a set of variables X is recursively defined by the following grammar:

$$\begin{array}{ll} t ::= x & \text{variable} \\ \quad | f(t_1, \dots, t_n) & f \text{ of arity } n \end{array} \quad (6)$$

- $T_{\Sigma}(X)$
- terms without variables (from $T_{\Sigma}(\emptyset)$ or short T_{Σ}): **ground** terms



Syntax: formulas of first-order logics

- given: first-order signature Σ

Definition (FO formulas)

The *first-order formulas* w.r.t. a signature Σ and a set of variables X is recursively defined by the following grammar, with $P \in \Sigma_{rel}$, $ar(P) = k$, $t_i \in T_{\Sigma}(X)$ for all $i \in \mathbb{N}$:

$$\begin{array}{lcl} \varphi & ::= & P(t, \dots, t) \mid t_1 \doteq t_2 \mid \top \mid \perp \quad \text{atomic formulas} \\ & & \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg \varphi \mid \varphi \rightarrow \varphi \mid \\ & & \mid \forall x. \varphi \mid \exists x. \varphi \end{array}$$



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Free and bound occurrences of variables

- quantifiers *bind* variables
- *scope*
- cf. binding or scoping mechanisms for programming languages
- variables can *occur* free or not (= *bound*) in a formula
- careful with substitution

How can we define “free variables”?



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Free and bound occurrences of variables

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- variables can *occur* free or not (= *bound*) in a formula
- careful with substitution

How can we define “free variables”?

Definition (Free variables)

Let Σ be a signature and X a set of variables. The set of *free variables* $fv(\varphi) \subseteq X$ of a FO formula φ is defined as:

$$fv(\varphi) = \begin{cases} \{x\} & \varphi = x \\ \{\} & \varphi \text{ is some other atomic formula (i.e. not a variable)} \\ fv(\psi) & \varphi = \neg\psi \\ fv(\psi) \cup fv(\chi) & \varphi = \psi \circ \chi \text{ with } \circ \in \{\wedge, \vee, \rightarrow\} \\ fv(\psi) \setminus \{x\} & \varphi = \forall x.\psi \text{ or } \varphi = \exists x.\psi \end{cases}$$



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Exercise: free variables



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What are the free variables in the following formulas?

- $\forall x.P(a)$
- $\forall x.Q(x, x) \rightarrow \exists x.Q(x, y)$
- $\forall x.(P(y) \wedge \forall y.P(x))$
- $\exists x.(\neg P(x) \vee P(f(a)))$
- $\exists x.\forall x.Q(x, f(x))$

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- **Substitution** = *replacement*, namely of variables by terms
- notation $t[s/x]$
- symbol θ
- take care of free and bound variables

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First-order structures and models

- given Σ
- assume single-sorted case

first-order structure (or model) $M = (A, I)$

- A some set (also called domain of the interpretation)
- **interpretation** I , respecting the arities of function symbols f and relational symbols P
 - $\llbracket f \rrbracket^I : A^n \rightarrow A$
 - $\llbracket P \rrbracket^I \subseteq A^n$



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Variable assignments: Giving meaning to variables



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Variable assignment

given Σ and model

$$\sigma : X \rightarrow A$$

alternative names

- valuation
- state

Extension/update

For $x \in X$ and $a \in A$ we define $\sigma[x \mapsto a] : X \rightarrow A$ as

$$(\sigma[x \mapsto a])(y) = \begin{cases} a & y = x \\ \sigma(y) & \text{else} \end{cases}$$

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(E)valuation of terms



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- σ “straightforwardly extended/lifted to terms”

How to define σ for terms?

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Definition

Given a signature Σ and a corresponding model $M = (A, I)$, the value of term t from $T_\Sigma(X)$, with variable assignment $\sigma : X \rightarrow A$ written $\llbracket t \rrbracket_\sigma^I$ is given inductively as follow:

$$\begin{aligned}\llbracket x \rrbracket_\sigma^I &= \sigma(x) \\ \llbracket f(t_1, \dots, t_n) \rrbracket_\sigma^I &= \llbracket f \rrbracket^I(\llbracket t_1 \rrbracket_\sigma^I, \dots, \llbracket t_n \rrbracket_\sigma^I) .\end{aligned}\tag{7}$$

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Example: Variable assignment



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- Let's consider the domain of natural numbers with the usual interpretation of the function symbols. Let id be the identity function ($\text{id} : x \mapsto x$ for all $x \in \mathbb{N}$), inc be the increment function and 1 a constant.

How are the terms evaluated?

- $\text{id}[x \mapsto 7][y \mapsto 2](+ (x, \text{inc}(y)))$
- $\text{id}[x \mapsto 7][y \mapsto 2][x \mapsto 2](+ (x, \text{inc}(y)))$
- $\text{id}[y \mapsto 2](+ (x, \text{inc}(1)))$

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Definition ($\models$)

$$M, \sigma \models \varphi$$

- Σ fixed
- in model M and with variable assignment σ formula φ is true (holds)
- M and σ satisfy φ
- minority terminology: M, σ model of φ
- one also finds notations like $\sigma \models_M \varphi$ (“in model M , σ satisfies formula φ ”)

$\models$ for FOL in a nutshell

$\models$ for propositional logic ($\wedge$ is “and” ...) + using the interpretation for terms and relations + using a variable assignment for free variables + defining $\exists$ and $\forall$ appropriately ($\exists x$ means “there exists a value for x ” ...)



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Definition ($\models$)

Let Σ be a signature, $M = (A, I)$ a corresponding FO-structure and $\sigma : X \rightarrow A$ a variable assignment. We define that M and σ **satisfy** φ ($M, \sigma \models \varphi$) as:

$M, \sigma \models \top$	
$M, \sigma \not\models \perp$	
$M, \sigma \models P(t_1, \dots, t_k)$	$(\sigma(t_1), \dots, \sigma(t_k)) \in \llbracket P \rrbracket^I,$ $P \in \Sigma_{rel}, ar(P) = k$ and $t_1, \dots, t_k \in T_\Sigma(X)$
$M, \sigma \models t_1 \doteq t_2$	$\sigma(t_1) = \sigma(t_2)$ and $t_1, t_2 \in T_\Sigma(X)$
$M, \sigma \models \neg \varphi$	$M, \sigma \not\models \varphi$
$M, \sigma \models \varphi \wedge \psi$	$M, \sigma \models \varphi$ and $M, \sigma \models \psi$
$M, \sigma \models \varphi \vee \psi$	$M, \sigma \models \varphi$ or $M, \sigma \models \psi$
$M, \sigma \models \varphi \rightarrow \psi$	$M, \sigma \not\models \varphi$ or $M, \sigma \models \psi$
$M, \sigma \models \exists x. \varphi$	there exists $a \in A$ such that $M, \sigma[x \mapsto a] \models \varphi$
$M, \sigma \models \forall x. \varphi$	$M, \sigma[x \mapsto a] \models \varphi$ holds for all $a \in A$



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Interesting problems



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Let's consider only formulas φ with $fv(\varphi) = \emptyset$ for simplicity.

model checking

$$M \stackrel{?}{\models} \varphi$$

(decidable)

satisfiability

$$? \models \varphi$$

(undecidable)

validity

$$\stackrel{?}{\models} \varphi$$

(undecidable)

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Conclusion

- Propositional logic
 - very simple
 - foundation of all logics that we are going to consider
- First-order logic
 - way more powerful
 - properties of formula often undecidable
- We are going to consider **temporal logic** next



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- **Modal** logic: logic of “*necessity*” and “*possibility*”, in that originally the intended meaning of the *modal* operators $\Box$ and $\Diamond$ was
 - $\Box\varphi$: φ is necessarily true.
 - $\Diamond\varphi$: φ is possibly true.
- Depending on what we intend to capture: we can interpret $\Box\varphi$ differently.
 - temporal** φ will always hold in the future.
 - doxastic** I believe φ .
 - epistemic** I know φ .
 - intuitionistic** φ is provable.
 - deontic** It ought to be the case that φ .

We will restrict here² the modal operators to $\Box$ and $\Diamond$ (and mostly work with a temporal “mind-set”).

²For later chapters about various temporal models, there will be more.

Definition (Kripke frame and Kripke model)

- A *Kripke frame* is a structure (W, R) where
 - W is a non-empty set of *worlds*, and
 - $R \subseteq W \times W$ is called the *accessibility relation* between worlds.
- A **Kripke model** M is a structure (W, R, L) where
 - (W, R) is a frame, and
 - L a function of type $L : W \rightarrow (AP \rightarrow \mathbb{B})$, called *valuation*.
- isomorphically: $V : W \rightarrow 2^{AP}$

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Terminology

- Kripke structure is classical terminology (from modal logics)
- we work later with analogous structures: **(labelled) transition systems** and **automata**
- different terminology
 - worlds w vs. states $s \in S$ or $q \in Q$
 - accessibility (relation) vs. transition (relation) $\rightarrow$
- fingerprint applies:
 - are transitions labelled or not (and if so, with what)?
 - are states labelled or not (and if so, with what)?
 - do I have (or want to include) initial states in my definition?
 - does my “construct” have a notion of **acceptance**?
 - what’s the interpretation?



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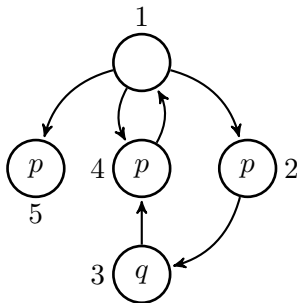
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- $W = \{w_1, w_2, w_3, w_4, w_5\}$
- $R = \{(w_1, w_5), (w_1, w_4), (w_4, w_1), \dots\}$
- $L = [w_1 \mapsto \emptyset, w_2 \mapsto \{p\}, w_3 \mapsto \{q\}, \dots]$

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Modalities "Box" and "Diamond"



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- modal operators $\Box$ and $\Diamond$
- often pronounced "**nessessarily**" and "**possibly**"
- mental picture: depends on the **kind** of logic (temporal, epistemic, deontic ..) and, related to that, the form of the **accessibility** relation R
- formal definition: see next slide

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the only interesting thing



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the only interesting thing

interpretation of $\Box$ and $\Diamond$

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Definition (Satisfaction)

A modal formula φ is **true** (or it **holds**) in the world w of a model $M = (W, R, L)$, written $M, w \models \varphi$, if:

$$M, w \models p \quad \text{iff} \quad L(w)(p) = \top$$

$$M, w \models \neg\varphi \quad \text{iff} \quad M, w \not\models \varphi$$

$$M, w \models \varphi_1 \vee \varphi_2 \quad \text{iff} \quad M, w \models \varphi_1 \text{ or } M, w \models \varphi_2$$

$$M, w \models \Box\varphi \quad \text{iff}$$

$$M, w \models \Diamond\varphi \quad \text{iff}$$



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$$M, w \models \varphi_1 \vee \varphi_2 \quad \text{iff} \quad M, w \models \varphi_1 \text{ or } M, w \models \varphi_2$$

$$M, w \models \Box\varphi \quad \text{iff} \quad M, w' \models \varphi, \text{ for all } w' \text{ such that } w R w'$$

$$M, w \models \Diamond\varphi \quad \text{iff} \quad M, w' \models \varphi, \text{ for some } w' \text{ such that } w R w'$$

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Different kinds of relations

R a *binary relation* on W , i.e., $R \subseteq W \times W$

- reflexive
- transitive
- (right) Euclidian
- total
- order relation
- . . .



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Valid in a frame/for a set or class of frames

If $(W, R, L), w \models \varphi$ for all w and L , we write

$$(W, R) \models \varphi$$

Example (Samples)

- $(W, R) \models \Box\varphi \rightarrow \varphi$ iff R is reflexive.
- $(W, R) \models \Box\varphi \rightarrow \Diamond\varphi$ iff R is total.
- $(W, R) \models \Box\varphi \rightarrow \Box\Box\varphi$ iff R is transitive.
- $(W, R) \models \neg\Box\varphi \rightarrow \Box\neg\Box\varphi$ iff R is Euclidean.



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Some exercises

Prove the double implications from the previous slide (i.e., think it through...)!



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Base line axiomatic system (“K”) (normal modal logics)



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φ is a propositional tautology
————— PL

φ

$\varphi \rightarrow \psi$ φ
————— MP
 ψ

————— K
 $\Box(\varphi_1 \rightarrow \varphi_2) \rightarrow (\Box\varphi_1 \rightarrow \Box\varphi_2)$

φ
————— NEC
 $\Box\varphi$

Sample axioms for different accessibility relations



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$$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi) \quad (\text{K})$$

$$\Box\varphi \rightarrow \Diamond\varphi \quad (\text{D})$$

$$\Box\varphi \rightarrow \varphi \quad (\text{T})$$

$$\Box\varphi \rightarrow \Box\Box\varphi \quad (4)$$

$$\neg\Box\varphi \rightarrow \Box\neg\Box\varphi \quad (5)$$

$$\Box(\Box\varphi \rightarrow \psi) \rightarrow \Box(\Box\psi \rightarrow \varphi) \quad (3)$$

$$\Box(\Box(\varphi \rightarrow \Box\varphi) \rightarrow \varphi) \rightarrow (\Diamond\Box\varphi \rightarrow \varphi) \quad (\text{Dum})$$

Different “flavors” of modal logic



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Logic	Axioms	Interpretation	Properties of R
D	K D	deontic	total
T	K T		reflexive
K45	K 4 5	doxastic	transitive/euclidean
S4	K T 4		reflexive/transitive
S5	K T 5	epistemic	reflexive/euclidean
			reflexive/symmetric/transitive equivalence relation

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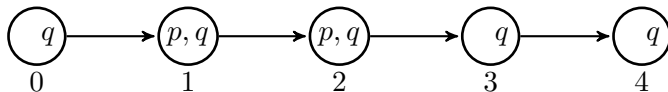
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Some exercises

Consider the frame (W, R) with $W = \{1, 2, 3, 4, 5\}$ and $(i, i + 1) \in R$



- $M, 0 \models \Diamond \Box p$
- $M, 0 \models \Diamond \Box p \rightarrow p$
- $M, 2 \models \Diamond(q \wedge \neg p) \wedge \Box(q \wedge \neg p)$
- $M, 0 \models q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond q)))$
- $M \models \Box q$



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Exercises (3): validities



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Which of the following are *valid* in modal logic. For those that are not, argue why and find a class of frames on which they become valid.

1. $\Box \perp$
2. $\Diamond p \rightarrow \Box p$
3. $p \rightarrow \Box \Diamond p$
4. $\Diamond \Box p \rightarrow \Box \Diamond p$

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[Exercises](#)

Multi-modal logic

Syntax

The formulas of a multi-modal logic are given by the following grammar.

$$\varphi ::= p \mid \perp \mid \neg\varphi \mid \varphi \rightarrow \varphi \mid \Diamond_0\varphi \mid \Diamond_1\varphi \mid \Diamond_3\varphi \mid \dots \quad (8)$$

where p is from a set of propositional atoms.



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where p is from a set of propositional atoms.

“Kripke frame” $(W, R_0, R_1, \dots)$, where R_i are relations over W .

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where p is from a set of propositional atoms.

“Kripke frame” $(W, R_0, R_1, \dots)$, where R_i are relations over W .

Semantics

“natural” generalization of the “mono”-case

$$M, w \models \Diamond_a\varphi \text{ iff } \exists w' : wR_a w' \text{ and } M, w' \models \varphi \quad (9)$$

analogously for modality $\Diamond_b$ and relation R_b

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Remarks

- *multi-modal*: obvious generalization of modal logic
- relations can overlap; i.e., their intersection need not be empty
- *infinitely* many relations and infinitely many modalities possible
- often: labels $a, b, \dots$ or similar from a label set or alphabet Σ
- Cf. later: **labelled** transition systems

$$[a]\varphi \quad \text{and} \quad R_a$$

Notation: instead of $(w_1, w_2) \in R_a$ or $w_1 R_a w_2$

$$w_1 \xrightarrow{a} w_2$$



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Making Kripke structures “multi-modal-prepared”



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Definition (Labelled Kripke structures)

Assume a set of labels Σ . A *labelled Kripke structure* is a tuple (W, R, Σ) where

$$R = \bigcup_{l \in \Sigma} R_l \quad (10)$$

is the union of the relations indexed by the labels of Σ .

for us (at least now): The labels of Σ can be thought as programs

- Σ : aka alphabet,
- alternative: $R \subseteq W \times \Sigma \times W$
- labels $l, l_1 \dots$ but also $a, b, \dots$ or others
- often: $\xrightarrow{a}$, like $w_1 \xrightarrow{a} w_2$ or $s_1 \xrightarrow{a} s_2$

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