# Course Notes on Computable Analysis

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## Foreword

UNDER CONSTRUCTION ..... Thanks to Emil Haugen and ... Eirik Ergon Aung .... Erik Snilsberg ..... Juvenal Murwanashyaka .....

ABERTH: These notes are under construction. They are not complete, and they will be gradually extended throughout the foreseeable future. Until I regard the notes as reasonably self-contained, I will give some references to details in Aberth [1]. You find these references in paragraphs like this, that is, paragraphs labeled "ABERTH". ◇

The first three chapters are by now fairly self-contained, but beware of mispxrints (and please report any misprint you find to the author).

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# Chapter 1

# Preliminaries

## 1.1 General Stuff

UNDER CONSTRUCTION ..... UNDER CONSTRUCTION

The reader should be familiar with basic computability theory and Kleene-recursion. Section 1.2 gives a summary of what the reader needs to know. We will stick to the notation and definitions given in Chapter 7 of L&K [8] with one conspicuous exception: we will write $\Phi_e^n$ place of $\{e\}^n$ and accordingly $\Phi_e$ place of $\{e\}$.

## 1.2 Computability Theory

We define the computable functions by Kleene-recursion, also known as $\mu$-recursion.

**Definition 1.1.** *We define the set of* computable functions *inductively by the following six clauses.*

*(1) the successor function $\mathcal{S}$ is a computable function*

*(2) the projection function $\mathcal{I}_i^n$ is a computable function (for all $i, n \in \mathbb{N}$ such that $1 \leq i \leq n$)*

*(3) the zero function $\mathcal{O}$ is a computable function*

*(4) $f$ is computable function if*

$$f(x_1, \ldots, x_n) = h(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)) \qquad \text{(Composition)}$$

*and $g_1, \ldots, g_m$, and $h$ are computable functions (for all $m, n \in \mathbb{N}$)*

*(5) $f$ is computable function if*

$$\begin{aligned} f(x_1, \ldots, x_n, 0) &= g(x_1, \ldots, x_n) \\ f(x_1, \ldots, x_n, y+1) &= h(x_1, \ldots, x_n, y, f(x_1, \ldots, x_n, y)) \end{aligned}$$

(Primitive Recursion)

*and $g$ and $h$ are computable functions (for all $n \in \mathbb{N}$)*

*(6) $f$ is computable function if*

$$f(x_1, \ldots, x_n) = (\mu i)[g(x_1, \ldots, x_n, i)] \qquad \text{(Minimalization)}$$

*and $g$ is a computable function (for all $n \in \mathbb{N}$).*

*The set of* primitive recursive functions *is defined inductively as the set of computable functions, but without Clause (6).*

A function $f : \mathbb{N}^n \to \mathbb{N}$ is computable by a Turing machine if and only if it is computable according to Definition 1.1. All primitive recursive functions are total (a Turing machine that computes a primitive recursive function will terminate on every input), and we may now and then assume some acquaintance with primitive recursion. The class of primitive recursive functions has strong closure properties and is e.g., closed under bounded sums, bounded products and bounded minimalization that is, the definition scheme

$$f(x_1, \ldots, x_n, y) = (\mu i < y)[g(x_1, \ldots, x_n, i)]\,. \qquad \text{(Bounded Minimalization)}$$

**Definition 1.2.** *Let $A$ be a set of natural numbers or tuples of natural numbers, that is, $A \subseteq \mathbb{N}^n$ for some $n \geq 1$. We define the* characteristic function $\chi_A$ of $A$, *by*

$$\chi_A(x_1, \ldots, x_n) = \begin{cases} 0 & \text{if } (x_1, \ldots, x_n) \in A \\ 1 & \text{otherwise.} \end{cases}$$

$A$ set is computable *when the characteristic function of the set is computable.* $A$ set is primitive recursive *when the characteristic function of the set is primitive recursive.*

*We define the* domain *of the computable function* $f : \mathbb{N} \to \mathbb{N}$*, written* $\mathrm{dom}(f)$*, by*

$$\mathrm{dom}(f) \;=\; \{\; x \mid \textit{there exists } y \in \mathbb{N} \textit{ such that } f(x) = y \;\}\,.$$

*A set* $A$ *of natural numbers is* semi-computable *if there exists a computable function* $f$ *such that* $A = \mathrm{dom}(f)$*.*

*We define the* range *of the computable function* $f : \mathbb{N} \to \mathbb{N}$*, written* $\mathrm{rng}(f)$*, by*

$$\mathrm{rng}(f) \;=\; \{\; y \mid \textit{there exists } x \in \mathbb{N} \textit{ such that } f(x) = y \;\}\,.$$

*A set* $A$ *of natural numbers is* computably enumerable *if*

- $A$ *is finite*
- *or, there exists a total computable one-to-one function* $f$ *such that* $A = \mathrm{rng}(f)$*.*

Let $A \subset \mathbb{N}$. We will use $\overline{A}$ to denote the complement of $A$, and the reader should familiar with basic facts like

- $A$ is computable if and only if both $A$ and $\overline{A}$ are semi-computable
- $A$ is semi-computable if and only if $A$ is computably enumerable.

Relations and predicates are treated as sets: formally, an $n$-ary relation is nothing but a set of $n$-tuples, and a predicate is just an another name for a relation. In the days when computability theory was called recursion theory, computable sets were called *recursive sets*, and computably enumerable sets were called *recursively enumerable sets*. Moreover, *recursively enumerable* was abbreviated *r.e.* Hence, an r.e. set is nothing but a computably enumerable set. Many authors do still stick to the old terminology and prefer to talk about recursive and recursively enumerable sets. Besides, computable sets are also often called *decidable* sets, and accordingly semi-computable sets are called *semi-decidable* sets. The terminology varies from book to book, from author to author.

We assume some coding of the finite sequences of natural numbers into the natural numbers (any reasonable coding convention will work), and we use $\langle a_1, \ldots, a_n \rangle$ denote the code number of the sequence $a_1, \ldots, a_n$.

**Definition 1.3.** *The natural number* $e$ *is a* computable index for the computable function $f$ *if one of the following clauses holds:*

*(1)* $e = \langle 0 \rangle$ *and* $f$ *is the successor function* $\mathcal{S}$

*(2)* $e = \langle 1, n, i \rangle$ *and* $f$ *is the projection function* $\mathcal{I}_i^n$

*(3)* $e = \langle 2 \rangle$ *and* $f$ *is the zero function* $\mathcal{O}$

*(4)* $e = \langle 3, n, b_1, \ldots b_m, a \rangle$ *and*

$$f(x_1, \ldots, x_n) \;=\; h(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$$

*and* $b_1, \ldots b_m, a$ *are indices for respectively* $g_1, \ldots, g_m, h$.

*(5)* $e = \langle 4, n, a, b \rangle$ *and*

- $f(x_1, \ldots, x_n, 0) = g(x_1, \ldots, x_n)$
- $f(x_1, \ldots, x_n, y+1) = h(x_1, \ldots, x_n, y, f(x_1, \ldots, x_n, y))$

*and* $a, b$ *are indices for respectively* $g, h$.

*(6)* $e = \langle 5, n, a \rangle$ *and*

$$f(x_1, \ldots, x_n) \;=\; (\mu i)[g(x_1, \ldots, x_n, i)]$$

*and* $a$ *is an index for* $g$.

**Theorem 1.4** (Kleene's Normal Form Theorem)**.** *Let* $n \in \mathbb{N}$. *There exist a primitive recursive predicate* $\mathcal{T}_n$ *and a primitive recursive function* $\mathcal{U}$ *such that*

- *if* $e$ *is not an index for an* $n$*-ary function, then* $\mathcal{T}_n(e, x_1, \ldots x_n, t)$ *does not hold (for all* $x_1, \ldots x_n, t \in \mathbb{N}$*)*
- *if* $e$ *is an index for the* $n$*-ary function* $f$*, then*

$$f(x_1, \ldots x_n) \text{ is defined} \;\Leftrightarrow\; \text{there exists } t \text{ such that } \mathcal{T}_n(e, x_1, \ldots x_n, t) \text{ holds}$$

- *whenever* $f(x_1, \ldots x_n)$ *is defined*

$$f(x_1, \ldots x_n) \;=\; \mathcal{U}((\mu t)[\mathcal{T}_n(e, x_1, \ldots x_n, t)])\,.$$

**Definition 1.5.** *For each $e \in \mathbb{N}$, we define the* $e$-th computable function *of arity $n$, written $\Phi_e^n$, by*

$$\Phi_e^n(x_1, \ldots, x_n) \;=\; \mathcal{U}((\mu t)[\mathcal{T}_n(e, x_1, \ldots x_n, t)])$$

*where the function $\mathcal{U}$ and predicate $\mathcal{T}_n$ are given by Kleene's Normal Form Theorem (Theorem 1.4).*

**Theorem 1.6** (Enumeration Theorem)**.** *The sequence $\Phi_0^n, \Phi_1^n, \Phi_2^n, \ldots$ is an enumeration of the computable functions of arity $n$ in the sense that*

*(1) for each $e \in \mathbb{N}$, the function $\Phi_e^n$ is computable*

*(2) for each computable function $f$ of arity $n$, there exists $e$ such that*

$$f(x_1, \ldots, x_n) \;=\; \Phi_e^n(x_1, \ldots, x_n)\,.$$

*More over, this enumeration is computable in the sense that*

*(3) there exists a computable function $g$ of arity $n+1$ such that*

$$g(e, x_1, \ldots, x_n) \;=\; \Phi_e^n(x_1, \ldots, x_n)\,.$$

**Theorem 1.7** (The S-m-n Theorem)**.** *Let $m, n \in \mathbb{N}$. There exists a primitive recursive function $S_n^m$ (of arity $n+1$) such that*

$$\Phi^m_{S_n^m(e,x_1,\ldots,x_n)}(y_1, \ldots y_m) \;=\; \Phi_e^{n+m}(x_1, \ldots x_n, y_1, \ldots y_m)\,.$$

We will write $\Phi_e$ in place of $\Phi_e^1$. Note that there for any computable index $d$ exists a computable index $e$ such that $\Phi_e(\langle x_1, \ldots, x_n\rangle) = \Phi_d(x_1, \ldots, x_n)$. Thus, for any $n$-ary computable function $f$ there exists a computable index $e$ such that $\Phi_e(\langle x_1, \ldots, x_n\rangle) = f(x_1, \ldots, x_n)$.

**Definition 1.8.** *We define the* $e$-th semi-computable set*, written $\mathcal{W}_e$, by $\mathcal{W}_e = \mathrm{dom}(\Phi_e)$. We define the set $\mathcal{K}$ by $\mathcal{K} = \{x \mid x \in \mathcal{W}_x\}$.*

The reader should know that the set $\mathcal{K}$ is not computable although it is semi-computable. The reader should even be able to prove that $\mathcal{K}$ is semi-computable, but not computable. Moreover, the reader should be familiar with the next theorem.

**Theorem 1.9** (Rice's Theorem)**.** *Let $\mathcal{A}$ a set of unary computable functions, and let $A$ be the index set of $\mathcal{A}$, that is, $A = \{e \mid \Phi_e \in \mathcal{A}\}$. Furthermore, assume that $\mathcal{A}$ is not empty, and assume that $\mathcal{A}$ does not contain all the computable functions. Then, $A$ is not a computable set.*

Proofs of the theorems above can be found in Chapter 7 of L&K [8]. For more on computability theory, see Kleene [6], Rodgers [12], Odifreddi [10] and Cooper [5].

## 1.3 Integers and Rational Numbers

We assume some coding of the integers ($\mathbb{Z}$) and the rational numbers ($\mathbb{Q}$) into the natural numbers ($\mathbb{N}$). We consider such a coding to be trivial. Therefore we allow for computable functions from natural numbers into rational numbers, from pairs of rational numbers into rational numbers, etc., with no further comment. Exactly how we code integers and rationals does not matter as long as we behave reasonably (we will not worry about computational complexity). Below you find some examples of reasonable coding conventions. More details on how the integers and the rational numbers can be coded as equivalence classes of natural numbers, can be found in Ødegaard [15].

**Example 1.10.** Let $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a computable bijection (one-to-on and onto), e.g., let

$$\langle x, y \rangle \;=\; x + \frac{(x+y)(x+y+1)}{2} \,.$$

Let $(\cdot)_L : \mathbb{N} \to \mathbb{N}$ and $(\cdot)_R : \mathbb{N} \to \mathbb{N}$ be such that

$$(\langle x, y \rangle)_L = x \quad \text{and } (\langle x, y \rangle)_R = y \,.$$

We may use the following convention to code integers: The natural number $x$ represents the integer $(x)_L - (x)_R$. Use $[\![x]\!]_{\mathbb{Z}}$ to denote the integer $(x)_L - (x)_R$, that is, $[\![x]\!]_{\mathbb{Z}} = (x)_L - (x)_R$. Hence, any natural number represents an integer, and each integer is represented by infinitely many natural numbers. We have

$$[\![\langle 1, 4 \rangle]\!]_{\mathbb{Z}} \;=\; (\langle 1, 4 \rangle)_L - (\langle 1, 4 \rangle)_R \;=\; 1 - 4 \;=\; -3 \,.$$

and

$$[\![\langle 6, 2 \rangle]\!]_{\mathbb{Z}} \;=\; (\langle 6, 2 \rangle)_L - (\langle 6, 2 \rangle)_R \;=\; 6 - 2 \;=\; 4 \,.$$

We also have $[\![\langle 100, 103\rangle]\!]_{\mathbb{Z}} = -3$ and $[\![\langle 96, 92\rangle]\!]_{\mathbb{Z}} = 4$. We may define the equality relation $=_{\mathbb{Z}}$ on the integers by

$$[\![x]\!]_{\mathbb{Z}} =_{\mathbb{Z}} [\![y]\!]_{\mathbb{Z}} \quad \Leftrightarrow \quad (x)_L + (y)_R = (y)_L + (x)_R \, .$$

□

**Example 1.11.** Let $\langle x, y\rangle = 2^x(2y+1) - 1$. Let

$$(z)_L \;=\; \mu x[\, (\; 2^x \mid z+1 \;) \;\wedge\; \neg(\; 2^{x+1} \mid z+1 \;)\,]$$

where the predicate $(u \,|\, v)$ holds iff $\exists w[u \times w = v]$. Let

$$(z)_R \;=\; \frac{((z+1)/2^{(z)_L}) - 1}{2} \, .$$

We have $(\langle x, y\rangle)_L = x$ and $(\langle x, y\rangle)_R = y$. Moreover, $\langle \cdot, \cdot \rangle$, $(\cdot)_L$, $(\cdot)_R$ are primitive recursive functions, and $\langle \cdot, \cdot \rangle$ is a bijection. □

**Example 1.12.** We may use the following convention to code rationals: The natural number $x$ represents the rational number $[\![(x)_L]\!]_{\mathbb{Z}}/((x)_R+1)$. We use $[\![x]\!]_{\mathbb{Q}}$ to denote the the rational number $[\![(x)_L]\!]_{\mathbb{Z}}/((x)_R+1)$, that is

$$[\![x]\!]_{\mathbb{Q}} \;=\; \frac{[\![(x)_L]\!]_{\mathbb{Z}}}{(x)_R + 1} \, .$$

Hence, any natural number represents a rational number, and each rational number is represented by infinitely many natural numbers. We have

$$[\![\langle\, \langle 0, 1\rangle, 7\,\rangle]\!]_{\mathbb{Q}} \;=\; \frac{[\![(\langle\, \langle 0, 1\rangle, 7\,\rangle)_L]\!]_{\mathbb{Z}}}{(\langle\, \langle 0, 1\rangle, 7\,\rangle)_R + 1} \;=\; \frac{[\![\langle 0, 1\rangle]\!]_{\mathbb{Z}}}{7+1} \;=\; -\frac{1}{8} \, .$$

We have

$$[\![0]\!]_{\mathbb{Q}} \;=\; \frac{[\![(0)_L]\!]_{\mathbb{Z}}}{(0)_R + 1} \;=\; \frac{((0)_L)_L - ((0)_L)_R}{0+1} \;=\; \frac{0}{1} \;=\; 0$$

since $\langle 0, 0\rangle = 0$. We may define multiplication of rationals $\times_{\mathbb{Q}}$ by

$$[\![x]\!]_{\mathbb{Q}} \times_{\mathbb{Q}} [\![y]\!]_{\mathbb{Q}} \;:=\; \langle\; [\![(x)_L]\!]_{\mathbb{Q}} \times_{\mathbb{Z}} [\![(y)_L]\!]_{\mathbb{Q}} \,,\, ((x)_R + 1)((y)_R + 1) \;\rangle$$

where $\times_{\mathbb{Z}}$ is the multiplication operator on integers. □

## 1.4 Exercises

**Exercise 1**

Given the examples above, define the multiplication operator $\times_{\mathbb{Z}}$ on the integers. Define the addition operator $+_{\mathbb{Q}}$ and the equality relation $=_{\mathbb{Q}}$ on the rationals.

**Exercise 2 (solution)**

Prove that the paring function given in Example 1.10 is an injection, that is, prove that we have $m = m'$ and $n = n'$ if $\langle m, n\rangle = \langle m', n'\rangle$.

**Exercise 3**

Prove that the paring function given in Example 1.10 is a bijection.

**Exercise 4 (solution)**

Prove that the decoding function $(\cdot)_L$ given in Example 1.10 is primitive recursive.

**Exercise 5 (solution)**

Verify that we have $(\langle x, y\rangle)_L = x$ and $(\langle x, y\rangle)_R = y$ in Example 1.11. Hint: Look at the prime factorisation of $z + 1$.

# Chapter 2

# The Field of Computable Real Numbers

## 2.1 The Computable Real Numbers

Our mathematical model of computation is Kleene recursion. In this model we use the natural numbers to represent mathematical objects like truth values, finite sequences of natural number, finite strings over the alphabet $\{a, b, c\}$, formulas of a formal language (Gödel numbers) and computable functions (computable indices). We can easily represent rational numbers in our model of computation. Rational number are finite entities, and by setting up coding conventions, each rational number can be represented by a natural number (see Section 1.3). What about the real numbers? Now we are talking about infinite entities. To what extent can we represent real numbers in our model of computation?

The general idea is to represent a real number $a$ by a computable function which yields rational approximations to $a$, such a function can again be represented by a computable index (and that is a natural number). A computable function that represent a real number will be called a realizer. The formal definition follows.

**Definition 2.1.** *A (total) computable function* $f : \mathbb{N} \to \mathbb{Q}$ *is a* realizer *if*

$$| f(n) - f(m) | \ \leq \ (n+1)^{-1} \ + \ (m+1)^{-1}$$

*for any* $n, m \in \mathbb{N}$.

For any $f : \mathbb{N} \to \mathbb{Q}$, we use $\underline{f(n)}$ to denote the closed interval

$$\left[\ f(n) - (n+1)^{-1}\ ,\ f(n) + (n+1)^{-1}\ \right]\ .$$

We will use $\underline{f(n)}_\ell$ to denote the left endpoint of the interval, that is, the rational $f(n) - (n+1)^{-1}$, and will use $\underline{f(n)}_r$ to denote the right endpoint for the interval, that is, the rational $f(n)+(n+1)^{-1}$. We will say that a rational $q$ *lies in the interval* $\underline{f(n)}$ when $\underline{f(n)}_\ell \leq q \leq \underline{f(n)}_r$. The notation $q \in \underline{f(n)}$ is simply a convenient way to state that the rational $q$ lies in the interval $\underline{f(n)}$, and the reader should note that

$$q \in \underline{f(n)} \quad \Leftrightarrow \quad |\, q - f(n)\,| \leq (n+1)^{-1}\ . \tag{2.1}$$

Finally, two interval *overlap* if there exists a rational that lies in both intervals.

We will use $\alpha, \beta, \gamma, \ldots$ to denote realizers. Definition 2.1 simply states that $\alpha : \mathbb{N} \to \mathbb{Q}$ is a realizer if the intervals $\underline{\alpha(n)}$ and $\underline{\alpha(m)}$ overlap for any $n, m \in \mathbb{N}$, and thus it seems like a good idea to restate Definition 2.1 as Definition 2.2 below.

**Definition 2.2.** *A (total) computable function $\alpha : \mathbb{N} \to \mathbb{Q}$ is a* realizer *if $\underline{\alpha(n)}$ and $\underline{\alpha(m)}$ overlap for any $n, m \in \mathbb{N}$.*

Note the similarity between the next definition and Definition 2.1.

**Definition 2.3.** *Two realizers $\alpha$ and $\beta$ are equivalent, written $\alpha \equiv \beta$, if*

$$|\alpha(n)\ -\ \beta(m)|\ \ \leq\ \ (n+1)^{-1}\ +\ (m+1)^{-1}$$

*for any $n, m \in \mathbb{N}$.*

Observation: $\alpha$ and $\beta$ are equivalent relaizers if and only if the intervals $\underline{\alpha(n)}$ and $\underline{\beta(m)}$ overlap for any $n, m \in \mathbb{N}$. Thus, the previous definition may be restated as the next definition.

**Definition 2.4.** *Two realizers $\alpha$ and $\beta$ are equivalent, written $\alpha \equiv \beta$, if $\underline{\alpha(n)}$ and $\underline{\beta(m)}$ overlap for any $n, m \in \mathbb{N}$.*

**Theorem 2.5.** *The relation $\equiv$ is an equivalence relation on the realizers.*

*Proof.* Any interval overlaps with itself. If if an interval $I_0$ overlaps with and interval $I_1$, then $I_1$ will also overlap with $I_0$. Thus, it is obvious that the relation is reflexive

and symmetric. We need to prove that the relation is transitive. To that end, assume $\alpha \equiv \beta$ and $\beta \equiv \gamma$ (we will prove $\alpha \equiv \gamma$).

Assume for the sake of a contradiction that $\alpha \not\equiv \gamma$. Then we have $n, m$ such that $\underline{\alpha(n)}$ and $\underline{\gamma(m)}$ do not overlap. We can w.l.o.g. assume that $\underline{\alpha(n)}_r < \underline{\gamma(m)}_\ell$. Pick a natural number $k$ such that $2/(k+1) < \underline{\gamma(m)}_\ell - \underline{\alpha(n)}_r$. Then, $\underline{\beta(k)}$ cannot overlap with both $\underline{\alpha(n)}$ and $\underline{\gamma(m)}$ as the length of $\underline{\beta(k)}$ is $2(k+1)$. Hence we have $k, n, m$ such that

$$\underline{\alpha(n)} \text{ and } \underline{\beta(k)} \text{ do not overlap}$$

or

$$\underline{\beta(k)} \text{ and } \underline{\gamma(m)} \text{ do not overlap.}$$

Hence, $\alpha \not\equiv \beta$ or $\beta \not\equiv \gamma$. This contradicts that $\alpha \equiv \beta$ and $\beta \equiv \gamma$. □

**Definition 2.6.** *For any realizer $\alpha$, we define the equivalence class $\{\alpha\}$ by*

$$\{\alpha\} \;=\; \{\; \beta \mid \beta \equiv \alpha \;\}\,.$$

*We define the set $\mathbb{R}_c$ by*

$$\mathbb{R}_c \;=\; \{\, \{\alpha\} \mid \alpha \textit{ is a realizer} \}\,.$$

*The elements of $\mathbb{R}_c$ are* the computable real numbers*, and we will use $a, b, c, \ldots$ to denote computable real numbers. If $a = \{\alpha\}$, we will say that $\alpha$* is a realizer of *$a$ and that $\alpha$* realizes *$a$. We will use $\mathbb{R}$ to denoted the (classic) real numbers.*

Before we go on and define operators on the computable reals, let us reflect upon what we have accomplished so far. We have defined the computable real numbers (the set of numbers we denote $\mathbb{R}_c$). We have not defined the real numbers (the set of numbers we denote $\mathbb{R}$). Our definition of the computable real numbers does not refer in any way to the definition of the real numbers. Some might find this philosophically interseting. Some might find this mathematically interesting. And some might just find this annoying and inconvenient (I have met a few). Be that as it may. Our development of computable analysis will be a rather standard one, and that's good enough for us.

For sure, we could have given or presupposed some definition of the real numbers, and then, for example, defined a real number $a$ to be a computable real number if there exist a computable function $f : \mathbb{N} \to \mathbb{Q}$ such that $|\, a - f(n)| \leq (n+1)^{-1}$ holds for any $n \in \mathbb{N}$. That should have worked fine, and if we had done so, the theory we

are about to develop would have amounted to more or less the same, at least from mathematical point of view. But we did not proceed like that. To us a computable real number is an equivalence class of a certain kind of computable functions which we call realizers. Any realizer $f$ yields a computable sequence $\underline{f(0)}, \underline{f(1)}, \underline{f(2)}, \ldots$ of closed intervals with rational endpoints. All the intervall in the sequence overlap, and as the sequence proceeds, the length of those intervals will become arbitrarily short, but never zero. It is natural, and heuristically helpful, to think that there exists one, and only one, real number $a$ which will be contained in all these intervalls, but formally the object $a$ does not exists. According to our definitions, all that is there is the computable sequence of intervals, and if you see something more, you see a ghost, but as long you remember that ghosts do not exist, it is likely to be a friendly and helpful ghost.

In spite of our formal definitions (which we stick to when prove our theorems), it does of course make perfectly good sense to claim that some familiar real number, like $e$ ($2.71828\ldots$) or $\pi$ ($3.14159\ldots$), is a computable real number. It also makes perfectly good sense to claim that there is a realizer for $e$. It is obvious what we intend to say if we make such a claim. We intend to say that there is a computable function $f : \mathbb{N} \to \mathbb{Q}$ such that $|e - f(n)| < (n+1)^{-1}$ holds for all $n$ (such an $f$ will be a realizer according to our definitions). Every rational will obviously be a computable real number. E.g, let $f(n) = 3/4$ for all $n \in \mathbb{N}$. Then $f$ is a realizer for the rational number $3/4$. Furthermore, it is not hard to see that irrationals like $e, \pi, \sqrt{2}, \cos(\frac{\pi}{4}), \ldots$ also will computable reals. These numbers are sums of well known series which easily can be converted to realizers. E.g., Euler's series $e = \sum_{i=0}^{\infty}(i!)^{-1}$ yields straightaway a realizer for $e$: let $f(n) = \sum_{i=0}^{n+1}(i!)^{-1}$. Then $f$ is a realizer for $e$. Indeed, all the familiar numbers and constants of the elementary calculus are computable real numbers.

## 2.2 Operators

We will now define the addition operator on the computable real numbers. The operator will be denoted $\hat{+}$. We will first define the operator $\hat{+}$. Thereafter we will prove that $\hat{+}$ together with a multiplication operator satisfies the axioms of a field, that is, we will prove that $\hat{+}$ actually behaves the way we expect it to behave.

Recall that a computable real number is an equivalence class of realizers. Thus, the operator $\hat{+}$ should work on such equivalence classes. We start by defining an

addition operator $(\alpha + \beta)$ that works on realizer $\alpha$ and $\beta$, and then we lift $(\alpha + \beta)$ to the operator $\hat{+}$ that works on equivalence classes of realizers.

**Definition 2.7.** *Let $\alpha$ and $\beta$ be realizers. We define the function $(\alpha + \beta)$ by*

$$(\alpha + \beta)(n) = \alpha(f_+(n)) + \beta(f_+(n)) \tag{2.2}$$

*where*

$$\begin{aligned} f_+(x) \;=\; (\mu i) \Big[ \Big| \; \alpha(i) + \beta(i) - (\underline{\alpha(i)}_\ell + \underline{\beta(i)}_\ell) \; \Big| \;\leq\; (x+1)^{-1} \quad \wedge \\ \Big| \; \alpha(i) + \beta(i) - (\underline{\alpha(i)}_\ell + \underline{\beta(i)}_r) \; \Big| \;\leq\; (x+1)^{-1} \quad \wedge \\ \Big| \; \alpha(i) + \beta(i) - (\underline{\alpha(i)}_r + \underline{\beta(i)}_\ell) \; \Big| \;\leq\; (x+1)^{-1} \quad \wedge \\ \Big| \; \alpha(i) + \beta(i) - (\underline{\alpha(i)}_r + \underline{\beta(i)}_r) \; \Big| \;\leq\; (x+1)^{-1} \Big] \;. \end{aligned} \tag{2.3}$$

Note that the first occurrence of the symbol + in (2.2) is simply a part of the name of a computable function, whereas the third occurrence is the addition operator on rational numbers. The addition operator on the rational numbers is a computable function.

**Lemma 2.8.** *Let $\alpha$ and $\beta$ be realizers. Then, for any $r_1, r_2 \in \mathbb{Q}$ and any $n \in \mathbb{N}$, we have*

$$r_1 \in \underline{\alpha(f_+(n))} \;\; \textit{and} \;\; r_2 \in \underline{\beta(f_+(n))} \quad \Rightarrow \quad r_1 + r_2 \in \underline{(\alpha + \beta)(n)} \;.$$

*Proof.* This follows in some sense rather straightforwardly from the definition of $(\alpha + \beta)$. Indeed, $f_+(n)$ is more or less defined to be the least $i$ such that

$$r_1 \in \underline{\alpha(i)} \;\text{ and }\; r_2 \in \underline{\beta(i)} \quad \Rightarrow \quad r_1 + r_2 \in \underline{(\alpha + \beta)(n)}$$

but still, it might be interesting and clarifying to see a meticulous proof.

First we will argue that $f_+(x) = 2x + 1$. Definition 2.7 says that $f_+(x)$ equals the least $i$ such that the conjunction of four inequalities holds. The first conjunct says that $i$ should be such that

$$\Big| \; \alpha(i) + \beta(i) - (\underline{\alpha(i)}_\ell + \underline{\beta(i)}_\ell) \; \Big| \;\leq\; (x+1)^{-1} \;.$$

We have

$$\begin{aligned} \alpha(i) + \beta(i) - (\underline{\alpha(i)}_\ell + \underline{\beta(i)}_\ell) \;= \\ \alpha(i) + \beta(i) - (\alpha(i) - (i+1)^{-1} + \beta(i) - (i+1)^{-1}) \;=\; 2(i+1)^{-1} \end{aligned}$$

and the least $i$ such that $2(i+1)^{-1} \leq (x+1)^{-1}$ is $2x+1$. If we also examine the three remaining conjuncts, we will see that we indeed have $f_+(x) = 2x+1$.

Now, assume that

$$r_1 \in \underline{\alpha(f_+(n))} = \underline{\alpha(2n+1)} \text{ and } r_2 \in \underline{\beta(f_+(n))} = \underline{\beta(2n+1)} \tag{2.4}$$

(we will prove that $r_1 + r_2 \in \underline{(\alpha+\beta)(n)}$).

By (2.4) and (2.1), we have

$$|r_1 - \alpha(2n+1)| \leq (2n+2)^{-1} \text{ and } |r_2 - \beta(2n+1)| \leq (2n+2)^{-1} \tag{2.5}$$

Furthermore, we have

$$\begin{aligned}
|r_1 + r_2 - (\alpha+\beta)(n)| &= |r_1 + r_2 - (\alpha(f_+(n)) + \beta(f_+(n)))| && \text{(by Def. 2.7)} \\
&= |r_1 + r_2 - (\alpha(2n+1) + \beta(2n+1))| \\
&= |(r_1 - \alpha(2n+1)) + (r_2 - \beta(2n+1))| \\
&\leq |(r_1 - \alpha(2n+1))| + |(r_2 - \beta(2n+1))| \\
&\leq (2n+2)^{-1} + (2n+2)^{-1} && \text{(by (2.5))} \\
&= (n+1)^{-1} .
\end{aligned}$$

This proves that $|r_1 + r_2 - (\alpha+\beta)(n)| \leq (n+1)^{-1}$, and thus we have $r_1 + r_2 \in \underline{(\alpha+\beta)(n)}$ by (2.1). □

**Theorem 2.9.** *Let $\alpha$ and $\beta$ be realizers. Then, the function $(\alpha+\beta)$ is a realizer.*

*Proof.* We saw in the proof of Lemma 2.8 that the unbounded search used to define $f_+(n)$ will terminate for any $n$ (we have $f_+(n) = 2n+1$). Hence it is obvious that $(\alpha+\beta)$ is a total computable function. We are left to prove that $(\alpha+\beta)$ satisfies Definition 2.2, that is, we must prove that $\underline{(\alpha+\beta)(n)}$ and $\underline{(\alpha+\beta)(m)}$ overlap for any $n, m$.

Pick arbitrary natural numbers $n$ and $m$. Now, $\underline{\alpha(f_+(n))}$ and $\underline{\alpha(f_+(m))}$ overlap since $\alpha$ is a realizer, and thus there will be a rational $r_1$ that lies in both intervals. By the same token there will be a rational number $r_2$ that lies in both $\underline{\beta(f_+(n))}$ and $\underline{\beta(f_+(m))}$. Thus, we have rationals $r_1, r_2$ such that

$$r_1 \in \underline{\alpha(f_+(n))} \text{ and } r_2 \in \underline{\beta(f_+(n))} \tag{2.6}$$

$$r_1 \in \underline{\alpha(f_+(m))} \text{ and } r_2 \in \underline{\beta(f_+(m))} \,. \tag{2.7}$$

By Lemma 2.8 and (2.6), we have $r_1 + r_2 \in \underline{(\alpha+\beta)(n)}$. By Lemma 2.8 and (2.7), we have $r_1 + r_2 \in \underline{(\alpha+\beta)(m)}$. Thus, $\underline{(\alpha+\beta)(n)}$ and $\underline{(\alpha+\beta)(m)}$ overlap. □

**Theorem 2.10.** *Let $\alpha, \alpha', \beta$ and $\beta'$ be realizers such that $\alpha \equiv \alpha'$ and $\beta \equiv \beta'$. Then $(\alpha+\beta) \equiv (\alpha'+\beta')$.*

*Proof.* We must prove that $\underline{(\alpha+\beta)(n)}$ and $\underline{(\alpha'+\beta')(m)}$ overlap for any $n, m$.

Pick arbitrary natural numbers $n$ and $m$. Since $\alpha$ and $\alpha'$ are equivalent realizers, $\underline{\alpha(f_+(n))}$ and $\underline{\alpha'(f_+(m))}$ overlap, and there will be a rational $r_1$ that lies in both intervals. Since $\beta$ and $\beta'$ are equivalent realizers, there will be a rational $r_2$ that lies in both $\underline{\beta(f_+(n))}$ and $\underline{\beta'(f_+(m))}$. Thus, we have rationals $r_1, r_2$ such that

$$r_1 \in \underline{\alpha(f_+(n))} \text{ and } r_2 \in \underline{\beta(f_+(n))} \tag{2.8}$$

$$r_1 \in \underline{\alpha'(f_+(m))} \text{ and } r_2 \in \underline{\beta'(f_+(m))} \,. \tag{2.9}$$

By Lemma 2.8 and (2.8), we have $r_1 + r_2 \in \underline{(\alpha+\beta)(n)}$. By (2.8) and (2.9), we have $r_1 + r_2 \in \underline{(\alpha'+\beta')(m)}$. Thus, $\underline{(\alpha+\beta)(n)}$ and $\underline{(\alpha'+\beta')(m)}$ overlap. □

Note the similarity between the proof of Theorem 2.9 and Theorem 2.10. It is a bit annoying that we have to repeat nearly the same proof. Can we avoid this? and if we can avoid this, at what cost?

Theorem 2.10 shows that the next definitions makes sense. By the theorem we can lift the operator $(\alpha+\beta)$ to an operator on equivalence classes of realizers, that is, to an oprator on computable real numbers.

**Definition 2.11.** *Let $\alpha$ and $\beta$ be any realizers. We define the addition operator $\hat{+}$ on the computable reals by*

$$\{\alpha\}\hat{+}\{\beta\} = \{(\alpha+\beta)\} \,.$$

Before we proceed and define multiplication and a few other operators on the computable reals, let us briefly discuss our definition of addition. During the proof of Lemma 2.8, we verified that we actually have

$$(\alpha+\beta)(n) = \alpha(2n+1) + \beta(2n+1) \,. \tag{2.10}$$

So why didn't we simply define $(\alpha + \beta)$ by (2.10)? Our official definition (2.7) of $(\alpha + \beta)$ is rather involved, and (2.10) yields a more compact and transparent definition. Moreover, by using (2.10), we could have given a more compact, and presumably also more transparent, proof of Theorem 2.9 (and thus also Theorem 2.10) as we have

$$\begin{aligned}
&|(\alpha+\beta)(n) - (\alpha+\beta)(m)| \\
&\qquad = \ |\alpha(2n+1) + \beta(2n+1) - (\alpha(2m+1) + \beta(2m+1))| \\
&\qquad = \ |(\alpha(2n+1) - \alpha(2m+1)) + (\beta(2n+1) - \beta(2m+1))| \\
&\qquad \leq \ |(\alpha(2n+1) - \alpha(2m+1))| \ + \ |(\beta(2n+1) - \beta(2m+1))| \\
&\qquad \leq \ (2n+2)^{-1} + (2m+2)^{-1} + (2n+2)^{-1} + (2m+2)^{-1} \\
&\qquad\qquad = \ (n+1)^{-1} + (m+1)^{-1} \qquad (2.11)
\end{aligned}$$

for any realizers $\alpha$ and $\beta$. From (2.11) (and Definition 2.1), we can conclude straightaway that $(\alpha + \beta)$ is a realizer whenever $\alpha$ and $\beta$ are realizers.

So why? Why did we proceed the way we did? The reason is that Definition 2.7 gives a useful template for defining operators on realizers. A few other important operators on realizers can be defined along the same lines, and moreover, proofs that these operators can be lifted to operators on the computable real numbers will follow the same lines as the proofs we have given above.[1] We will now restate Definition 2.7 and define a handful of other operators while we're at it.

**Definition 2.12.** *Let $\alpha$ and $\beta$ be realizers, and let $\circ$ be one of the rational operations $+, \times, -$ (addition, multiplication subtraction). We define the function $(\alpha \circ \beta)$ by*

$$(\alpha \circ \beta)(n) = \alpha(f_\circ(n)) \circ \beta(f_\circ(n)) \qquad (2.12)$$

*where*

$$\begin{aligned}
f_\circ(x) \ = \ (\mu i) \Big[ \ \Big| \ \alpha(i) \circ \beta(i) - (\underline{\alpha(i)}_\ell \circ \underline{\beta(i)}_\ell) \ \Big| \ \leq \ (x+1)^{-1} \quad \wedge \\
\Big| \ \alpha(i) \circ \beta(i) - (\underline{\alpha(i)}_\ell \circ \underline{\beta(i)}_r) \ \Big| \ \leq \ (x+1)^{-1} \quad \wedge \\
\Big| \ \alpha(i) \circ \beta(i) - (\underline{\alpha(i)}_r \circ \underline{\beta(i)}_\ell) \ \Big| \ \leq \ (x+1)^{-1} \quad \wedge \\
\Big| \ \alpha(i) \circ \beta(i) - (\underline{\alpha(i)}_r \circ \underline{\beta(i)}_r) \ \Big| \ \leq \ (x+1)^{-1} \ \Big] \ .
\end{aligned}$$

[1] Definition 2.7 is an environment friendly definition. It can be recycled and used over and over again to define various operators. The author's jokes are also environment friendly ... recycled and used over and over again.

*We use $\div$ to denote the division operator on rationals. We define the function $(\alpha \div \beta)$ by*

$$(\alpha \div \beta)(n) = \alpha(f_{\div}(n)) \div \beta(f_{\div}(n)) \tag{2.13}$$

*where*

$$f_{\div}(x) \;=\; (\mu i)\left[0 \notin \underline{\beta(i)} \;\wedge\; \ldots \textit{as above} \ldots\right] .$$

**Theorem 2.13.** *Let $\alpha$ and $\beta$ be realizers and let $\circ$ be one of the rational operations $+, \times, -$. Then, the function $(\alpha \circ \beta)$ is a realizer. Moreover, the function $(\alpha \div \beta)$ is a realizer if there exists $i$ such that $0 \notin \underline{\beta(i)}$.*

*Proof.* The proofs for the operators $\times, -$ are very similar to the proof for $+$ (see Theorem 2.9). We have to take some care when the operator is $\div$. □

**Theorem 2.14.** *Let $\circ$ be one let $\circ$ be one of the rational operations $+, \times, -$. Let $\alpha, \alpha', \beta$ and $\beta'$ be realizers such that $\alpha \equiv \alpha'$ and $\beta \equiv \beta'$. Then $(\alpha \circ \beta) \equiv (\alpha' \circ \beta')$, moreover, $(\alpha \div \beta) \equiv (\alpha' \div \beta')$ if there exists $i$ such that $0 \notin \underline{\beta(i)}$.*

*Proof.* The proofs for the operators $\times, -$ are very similar to the proof for $+$ (see Theorem 2.10). We have to take some care when the operator is $\div$. □

Theorem 2.14 guarantees that the next definition makes sense (we have already defined $\hat{+}$, but it doesn't make any harm to repeat the definition).

**Definition 2.15.** *Let $\alpha$ and $\beta$ be any realizers. We define the binary operators $\hat{+}$ and $\hat{\times}$ on the computable reals by*

$$\{\alpha\}\hat{+}\{\beta\} = \{(\alpha + \beta)\} \quad \textit{and} \quad \{\alpha\}\hat{\times}\{\beta\} = \{(\alpha \times \beta)\} .$$

*Moreover, for any rational number $r$, let $r^*$ denote the realizer given by $r^*(n) = r$ (for all $n \in \mathbb{N}$). We define the computable real numbers $\hat{0}$ and $\hat{1}$ by $\hat{0} = \{0^*\}$ and $\hat{1} = \{1^*\}$ (so we have $0^*(n) = 0$ and $1^*(n) = 1$ for all $n$).*

**Theorem 2.16** (Rice [11])**.** *$(\mathbb{R}_c, \hat{0}, \hat{1}, \hat{+}, \hat{\times})$ is a field, that is, the structure $(\mathbb{R}_c, \hat{0}, \hat{1}, \hat{+}, \hat{\times})$ is a model for the first-order axioms given in Figure 2.1.*

*Proof.* First we prove that the structure satisfies the axiom $(\forall xy)[x + y = y + x]$. We we will carry out a meticulous proof, step-by-step, we proceed as follows:

The Axioms of a Field

1. $(\forall xy)[x + y = y + x]$ (commutative law)
2. $(\forall xy)[x \times y = y \times x]$ (commutative law)
3. $(\forall xyz)[(x + y) + z = x + (y + z)]$ (associative law)
4. $(\forall xyz)[(x \times y) \times z = x \times (y \times z)]$ (associative law)
5. $(\forall xyz)[(x \times (y + z) = (x \times y) + (x \times z)]$ (distributive law)
6. $(\forall x)[x + 0 = x]$ (identity element)
7. $(\forall x)[x \times 1 = x]$ (identity element)
8. $(\forall x)(\exists y)[x + y = 0]$ (existence of inverse)
9. $(\forall x)[x \neq 0 \rightarrow (\exists y)[x \times y = 1]]$ (existence of inverse)

Figure 2.1: The axioms of a field.

- Let $a, b$ be arbitrary elements in $\mathbb{R}_c$. We will prove that prove $a\hat{+}b = b\hat{+}a$.
- By Definition 2.6 we have realizers $\alpha$ and $\beta$ such that $a = \{\alpha\}$ and $b = \{\beta\}$. Thus, we need to prove that $\{\alpha\}\hat{+}\{\beta\} = \{\beta\}\hat{+}\{\alpha\}$.
- Definition 2.15 states that $\{\alpha\}\hat{+}\{\beta\} = \{(\alpha + \beta)\}$ and $\{\beta\}\hat{+}\{\alpha\} = \{(\beta + \alpha)\}$. Thus, we need to prove that $\{\alpha + \beta\} = \{\beta + \alpha\}$.
- By Definition 2.6, we have $\{(\alpha + \beta)\} = \{(\beta + \alpha)\}$ iff $(\alpha + \beta) \equiv (\beta + \alpha)$. Thus, we need to prove that $(\alpha + \beta) \equiv (\beta + \alpha)$.
- By Definition 2.4, we have $(\alpha + \beta) \equiv (\beta + \alpha)$ iff the intervals $\underline{(\alpha + \beta)(n)}$ and $\underline{(\beta + \alpha)(m)}$ overlap for any $n, m$. Thus, we need to prove that the interval $\underline{(\alpha + \beta)(n)}$ and the interval $\underline{(\beta + \alpha)(m)}$ overlap for any $n, m$.

All the step we have carried out so far are just routine steps. We have simply unwinded definitions. *Here comes the essence of our proof:* We prove that the intervals $\underline{(\alpha + \beta)(n)}$ and $\underline{(\beta + \alpha)(m)}$ overlap for arbitrary natural numbers $n$ and $m$. Now, $\alpha$ and $\beta$ are realizers. By the definition of a realizer, that is Definition 2.2,

we have rationals $r_1, r_3$ such that

$$r_1 \in \underline{\alpha(f_+(n))} \text{ and } r_1 \in \underline{\alpha(f_+(m))} \,. \tag{2.14}$$

and

$$r_2 \in \underline{\beta(f_+(n))} \text{ and } r_2 \in \underline{\beta(f_+(m))} \,. \tag{2.15}$$

By (2.14), (2.15) and Lemma 2.8, we have

$$r_1 + r_2 \in \underline{(\alpha + \beta)(n)} \text{ and } r_2 + r_1 \in \underline{(\beta + \alpha)(m)} \,.$$

Now we use that *the commutative law of additions holds for the rational numbers*. We have $r_1 + r_2 = r_2 + r_1$, and thus the two interval overlap.

This proves that the structure $(\mathbb{R}_c, \hat{0}, \hat{1}, \hat{+}, \hat{\times})$ satisfies the commutative law of addition. The proofs that the structure satisfies the commutative law of multiplication, the associative laws and the distributive law are similar: Do the routine steps, that is, unwind some definitions, and the use the corresponding laws for rational numbers to prove that certain intervals overlap.

In order to prove that $\hat{0}$ is identity element for $\hat{+}$, we need to prove that $(\alpha + 0^*) \equiv \alpha$ holds for any realizer $\alpha$ (see exercise with soulution). In order to prove that $\hat{1}$ is identity element for $\hat{\times}$, we need to prove that $(\alpha \times 1^*) \equiv \alpha$ holds (for any realizer $\alpha$). In order to prove that addition has an inverse, we need to prove that $(\alpha + (0^* - \alpha)) \equiv 0^*$ holds (for any realizer $\alpha$). In order to prove that multiplication has an inverse, we need to prove that $(\alpha \times (1^* \div \alpha)) \equiv 1^*$ if $\alpha \not\equiv 0^*$. (for any realizer $\alpha$). □

## 2.3 A Little Bit on Notation

For pedagogical reasons, we have used the symbol $\hat{+}$ to denote the addition operator on the computable real numbers. If we had used the symbol $+$, we would have run the risk of causing unnecessary confusion. Maybe some readers would have confused the addition operator on the computable reals with the addition operator on the rational numbers which also is denoted $+$; or maybe with uninterpreted binary function symbol $+$ occurring in the first-order axioms in Figure 2.1. Careless and stupid readers are up to anything, but we will not help these poor people anymore. From now on we will also use $+$ to denote the addition operator on the computable reals since that is the standard notation. When you're in Rome do like the Romans.

By the same token, we will from now on write $0, 1, \times$ in place of $\hat{0}, \hat{1}, \hat{\times}$, respectively. Moreover, for any rational $r$, we will just write $r$ in place of $\{r^*\}$.

So from now on the reader has to figure out how to read expressions like $1+1$ on his[2] own. That might not be all that easy. Maybe the symbol $+$ refer to the operator we used to denote $\hat{+}$? Yes, that might be case. What's written there might be the expression we until recently would have written $\{1^*\}\hat{+}\{1^*\}$. Or maybe the symbol $+$ refer to the computable addition operator on the natural numbers? Or maybe we should consider 1 as an element of $\mathbb{R}$? In that case $+$ refer to the addition operators on the real numbers (an operator we have just taken for granted and never defined). Or maybe ...? There is a lot of options. Hopefully the context will tell the reader which one to pick.

Of course, this ambiguity-problem has been with us all time. Symbols like 0, 1, $+$, $\times$, $<$, $\leq$ are heavily overloaded in our corner of the world. But the problem is getting worse right here and now because we also want to use these symbols when we are working with computable real numbers. And the difference between, e.g., addition on rational numbers and addition on computable real numbers, matters. We are dealing with two operation of rather different nature, but also to operations that are easy to confuse.

## 2.4 Inequalities

We will now define the less-than and the less-than-or-equal-to ordering of the computable real numbers.

**Definition 2.17.** *Let $\alpha$ and $\beta$ be realizers. We define the binary relations $<, \leq$ over the computable reals by*

$$\{\alpha\} < \{\beta\} \;\Leftrightarrow\; \text{there exist } n, m \in \mathbb{N} \text{ such that } \underline{\alpha(n)}_r < \underline{\beta(m)}_\ell$$

*and*

$$\{\alpha\} \leq \{\beta\} \quad \Leftrightarrow \quad \{\alpha\} < \{\beta\} \text{ or } \{\alpha\} = \{\beta\}\ .$$

*Whenever convenient, we will write $\{\beta\} > \{\alpha\}$ and $\{\beta\} \geq \{\alpha\}$ in place of $\{\alpha\} < \{\beta\}$ and $\{\alpha\} \leq \{\beta\}$, respectively.*

[2]We assume that women do not exist.

Again, recall that the computable real numbers are equivalence classes of realizers. We should prove that our definition of $<$ really make sense as a relation between such equivalence classes, that is, we should prove Lemma 2.19 below, but first we will prove lemma which gives a helpful characterisation of the less-than relation.

**Lemma 2.18.** *For any realizers $\alpha, \beta$, we have $\{\alpha\} < \{\beta\}$ if and only if there for any $M \in \mathbb{N}$ exists $N \in \mathbb{N}$ such that*

$$\underline{\alpha(n)}_r + M(N+1)^{-1} \;<\; \underline{\beta(n)}_\ell$$

*holds for any $n \geq N$.*

*Proof.* We assume $\{\alpha\} < \{\beta\}$. By Definition 2.17, we have $n', m'$ such that $\underline{\alpha(n')}_r < \underline{\beta(m')}_\ell$. Let $M$ be any natural number. Pick $N$ such that

$$(N+1)^{-1} \;<\; (\underline{\beta(m')}_\ell - \underline{\alpha(n')}_r)/(M+4)$$

or, equivalently

$$(M+4)(N+1)^{-1} \;<\; (\underline{\beta(m')}_\ell - \underline{\alpha(n')}_r)$$

or, equivalently

$$\underline{\alpha(n')}_r + (M+4)(N+1)^{-1} \;<\; \underline{\beta(m')}_\ell \,.$$

Thus, the distance between the right endpoint of the interval $\underline{\alpha(n')}$ and the left endpoint of the intervall $\underline{\beta(m')}$ is strictly greater than $(M+4)(N+1)^{-1}$ whereas, for any $n \geq N$, the length of the interval $\underline{\alpha(n)}$, as well as the length of the intervall $\underline{\beta(n)}$, is shorter than $2(N+1)^{-1}$. Moreover, for any $n \geq N$, $\underline{\alpha(n)}$ overlaps with $\underline{\alpha(n')}$, and $\underline{\beta(n)}$ overlaps with $\underline{\beta(m')}$, and thus

$$\underline{\alpha(n)}_r + M(N+1)^{-1} \;<\; \underline{\beta(n)}_\ell \,.$$

This proves the left-right implication of the lemma. The right-left implication is trivial. □

**Lemma 2.19.** *Let $\alpha, \alpha', \beta, \beta'$ be realizers such that $\alpha \equiv \alpha'$ and $\beta \equiv \beta'$ Then we have*

$$\{\alpha\} < \{\beta\} \quad \Leftrightarrow \quad \{\alpha'\} < \{\beta'\} \,.$$

*Proof.* First we prove that $\{\alpha\} < \{\beta\}$ implies $\{\alpha'\} < \{\beta'\}$. We assume $\{\alpha\} < \{\beta\}$. By Lemma 2.18, we have $N \in \mathbb{N}$ such that

$$\underline{\alpha(n)}_r + 4(N+1)^{-1} \ < \ \underline{\beta(n)}_\ell$$

holds for any $n \geq N$. Fix some $k \geq N$. The interval $\underline{\alpha'(k)}$ is shorter than $2(N+1)^{-1}$ and overlaps with $\underline{\alpha(k)}$. The interval $\underline{\beta'(k)}$ is shorter than $2(N+1)^{-1}$ and overlaps with $\underline{\beta(k)}$. Hence we have $\underline{\alpha'(k)}_r < \underline{\beta'(k)}_\ell$. By Definition 2.17, we have $\{\alpha'\} < \{\beta'\}$. This proves that

$$\{\alpha\} < \{\beta\} \quad \Rightarrow \quad \{\alpha'\} < \{\beta'\} \ .$$

A symmetric argument yields

$$\{\alpha'\} < \{\beta'\} \quad \Rightarrow \quad \{\alpha\} < \{\beta\}$$

and thus the lemma holds. □

**Theorem 2.20.** *Let $\alpha$ and $\beta$ be any realizers. One, and only one, of the following three relations holds.*

$$(i) \ \{\alpha\} = \{\beta\} \qquad (ii) \ \{\alpha\} < \{\beta\} \qquad (iii) \ \{\beta\} < \{\alpha\} \ .$$

*Proof.* The proof splits into the two cases $\{\alpha\} = \{\beta\}$ and $\{\alpha\} \neq \{\beta\}$. (These two cases are mutually exclusive and exhaustive. We work in classical logic, and we regard the statement $\{\alpha\} \neq \{\beta\}$ as the negation of the statement $\{\alpha\} = \{\beta\}$.)

Assume $\{\alpha\} = \{\beta\}$. Then $\underline{\alpha(n)}$ and $\underline{\beta(m)}$ overlap for all natural numbers $n$ and $m$. Hence, we cannot have $n, m$ such that $\underline{\alpha(n)}_r < \underline{\beta(m)}_\ell$, and then $\{\alpha\} < \{\beta\}$ does not hold. Neither can we have $n, m$ such that $\underline{\beta(n)}_r < \underline{\alpha(m)}_\ell$, and then $\{\beta\} < \{\alpha\}$ does not hold. Hence, if (i) holds, neither (ii) nor (iii) will hold.

Assume $\{\alpha\} \neq \{\beta\}$. Then there exists $n, m$ such that $\underline{\alpha(n)}$ and $\underline{\beta(m)}$ do not overlap. Thus, at least one of

(a) $\underline{\alpha(n)}_r < \underline{\beta(m)}_\ell$

(b) $\underline{\beta(m)}_r < \underline{\alpha(n)}_\ell$

holds. Now, if both (a) and (b) holds, we have an obvious contradiction (we have $\underline{\alpha(n)}_\ell < \underline{\alpha(n)}_r < \underline{\beta(m)}_\ell < \underline{\beta(m)}_r < \underline{\alpha(n)}_\ell$). Thus, we conclude that exactly one of (a) and (b) holds, and then, exactly one of (ii) and (iii) holds. □

The Axioms of an Ordered Field

The first 9 axioms is given in Figure 2.1. Furthermore, we have the axioms ...

10. $(\forall xyz)[\ x < y\ \rightarrow\ x + z < y + z\ ]$
11. $\forall xyz)[\ x < y \wedge z > 0\ \rightarrow\ x \times z < y \times z\ ]$
12. $(\forall xyz)[\ x < y \wedge y < z\ \rightarrow\ x < z\ ]$
13. $(\forall xy)[\ x < y\ \vee\ x = y \vee y < x\ ]$

Figure 2.2: The axioms of an ordered field.

We have allready seen that the operators $+, \times$ on the computable real numbers satisfy the axioms of a field. Together with the relation $<$, they satisfy the axioms of and ordered field.

**Theorem 2.21.** *The opertors $+, \times$ together with the relation $<$ on the computable real numbers satisfy the axioms of an ordered field (see Figure 2.2).*

*Proof.* First we prove that the 10th axiom in Figure 2.2 is satisfied. Let $\alpha, \beta, \gamma$ be realizers, and assume that $\{\alpha\} < \{\beta\}$ (we need to prove that $\{(\alpha+\gamma)\} < \{(\beta+\gamma)\}$). By Lemma 2.18, we have $N$ such that

$$\underline{\alpha(k)}_r + 2(N+1)^{-1}\ <\ \underline{\beta(k)}_\ell$$

for all $k \geq N$, and a fortiori, we have

$$\alpha(k) + 2(N+1)^{-1}\ <\ \beta(k) \tag{2.16}$$

for all $k \geq N$. Furthermore, as $f_+(N) \geq N$, we have

$$\begin{aligned}
(\alpha+\gamma)(N) + 2(N+1)^{-1} &= \alpha(f_+(N)) + 2(N+1)^{-1} + \gamma(f_+(N)) && \text{(by Def. 2.7)} \\
&< \beta(f_+(N)) + \gamma(f_+(N)) && \text{(by (2.16))} \\
&= (\beta+\gamma)(N)\ . && \text{(by Def. 2.7)}
\end{aligned}$$

This proves that

$$(\alpha+\gamma)(N) + 2(N+1)^{-1}\ <\ (\beta+\gamma)(N)\ .$$

Recall that

$$\underline{(\alpha+\gamma)(N)}_r = (\alpha+\gamma)(N)+(N+1)^{-1} \text{ and } \underline{(\beta+\gamma)(N)}_\ell = (\beta+\gamma)(N)-(N+1)^{-1}.$$

Hence, we have $\underline{(\alpha+\gamma)(N)}_r < \underline{(\beta+\gamma)(N)}_\ell$, and then by Definition 2.17, we have $\{(\alpha+\gamma)\} < \{(\beta+\gamma)\}$. This shows that the 10th axiom is satisfied.

Next we prove that the 12th axiom is satisfied. Let $\alpha, \beta, \gamma$ be realizers, and assume $\{\alpha\} < \{\beta\}$ and $\{\beta\} < \{\gamma\}$ (we need to prove that $\{\alpha\} < \{\gamma\}$). By Lemma 2.18, we have $N_1, N_2$ such that

$$\underline{\alpha(k)}_r < \underline{\beta(k)}_\ell$$

for all $k \geq N_1$, and

$$\underline{\beta(k)}_r < \underline{\gamma(k)}_\ell$$

for all $k \geq N_2$. Let $N = \max(N_1, N_2)$. It is easy to se that $\underline{\alpha(N)}_r < \underline{\gamma(N)}_\ell$, and then we have $\{\beta\} < \{\gamma\}$ by Definition 2.17. This proves that the 12th axiom is satisfied.

The 13th axioms is satisfied by Theorem 2.20. We leave the 11th axioms as an exercise. □

## 2.5 Absolute Values

**Definition 2.22.** *Let $\alpha : \mathbb{N} \to \mathbb{N}$ be a computable function. We define the the function $|\alpha|$ by*

$$|\alpha|(n) = |\alpha(n)|.$$

**Theorem 2.23.** *The function $|\alpha|$ is a realizer if $\alpha$ is a realizer.*

*Proof.* Let $\alpha$ be a realizer. Now, since $\alpha$ is a realizer, the intervals $\underline{\alpha(n)}$ and $\underline{\alpha(m)}$ overlap for any $n, m \in \mathbb{N}$. It is easy to see that also the intervals $\underline{|\alpha(n)|}$ and $\underline{|\alpha(m)|}$ overlap for any $n, m \in \mathbb{N}$, and then, since $|\alpha|(n) = |\alpha(n)|$ and $|\alpha|(m) = |\alpha(m)|$, the intervals $\underline{|\alpha|(n)}$ and $\underline{|\alpha|(m)}$ overlap for any $n, m \in \mathbb{N}$. By Definition 2.4, $|\alpha|$ is a realizer. □

The previous theorem shows that the next definition makes sense.

**Definition 2.24.** *Let $\alpha$ any realizer. We define the unary operator $|\cdot|$ on the computable real numbers by $|\{\alpha\}| = \{|\alpha|\}$. We say that $|a|$ is the* absolute value *of the computable real number $a$.*

The next theorem tells that the operator we just have defined behaves as we expect. No surprises show up if one tries to prove the theorem.

**Theorem 2.25.** *For any $a, b \in \mathbb{R}_c$, we have*

- $|a| \geq 0$
- *if $a \geq 0$, then $|a| = a$*
- *if $a < 0$, then $|a| = -a$*
- $-|a| \leq a \leq |a|$
- $|a + b| \leq |a| + |b|$.

## 2.6 Decidability and Undecidability

We cannot decide if a computable index is an index for realizer. This is a straightforward corollary of Rice's Theorem (Theorem 1.9 in Section 1.2), but it might be instructive to see a direct proof.

**Claim (i)** The set $A = \{\, e \mid e \text{ is an index for a realizer} \,\}$ is undecidable.

Recall that $\mathcal{T}_1$ denote the Kleene $T$-predicate (see Theorem 1.4), and recall that $x \in \mathcal{K}$ iff there exists $t$ such that $\mathcal{T}_1(x, x, t)$ holds. Let $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ be the function given by

$$\psi(x, n) = \begin{cases} n & \text{if } (\exists t < n)[\mathcal{T}_1(x, x, t)] \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if $x \notin \mathcal{K}$, we have $\psi(x, n) = 0$ for all $n$; and if $x \in \mathcal{K}$, we have $\psi(x, n) = n$ for all $n$ sufficiently large $n$.

A quick technical remark: The reader should note that $\psi$ is a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{Q}$. Thus, when we, e.g., write $\psi(x, n) = n$, then the first occurrence of $n$ is the natural number $n$ where as the second occurrence is the rational number $n$. If we view $\psi$ as a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{N}$, then $\psi(x, n) = m$ where $m$ is the code number for the rational number $n$.

Anyway, it is obvious that $\psi$ is a (total) computable function, and hence, $\psi$ has a computable index $e$. By the S-m-n Theorem, we have

$$\psi(x,n) \;=\; \Phi_e(x,n) \;=\; \Phi_{S_1^1(e,x)}(n)\ .$$

Let $f(x) = S_1^1(e,x)$. Then $f$ is a primitive recursive function. If $x \notin \mathcal{K}$, we have $\Phi_{f(x)}(n) = 0$ for all $n$, and then $\Phi_{f(x)}(n)$ is a realizer (it obvious that $\Phi_{f(x)}(n)$ is a realizer for the computable real number 0). If $x \in \mathcal{K}$, we have $\psi(x,n) = n$, for all sufficiently large $n$, and then $\Phi_{f(x)}(n)$ is not a realizer. Hence, $f(x)$ is an index for a realizer iff $x \notin \mathcal{K}$.

Hence, we have a primitive recursive function such that $x \notin \mathcal{K}$ iff $f(x) \in A$. This proves that Claim (i) holds. (If we could decide membership in $A$, then we could also decide membership in $\mathcal{K}$: compute $f(x)$; $f$ is a primitive recursive function and thus the computation yields an output $m$; if $m \notin A$, then $x \in \mathcal{K}$; if $m \in A$, then $x \notin \mathcal{K}$.)

It is very interesting to study what we can decide, and cannot, decide in computable analysis, but we have to be a bit careful and ask the right question. In the realm of mathematics we are getting wiser by asking the right questions, not by answering the wrong ones.

Definition 1.2 states that a set of natural numbers is *computable* if the characteristic function of the set is computable. A decidable set is nothing but a computable set (the two words are used interchangeably). By Rice's Theorem, any sensible question related to realizers will be undecidable. Let $P$ be some property a realizer might, or might not, possess. Does a realizer $\alpha$ have the propery $P$? The question will be undecidable. Let $\mathcal{A}$ be the set of all realizer with property $P$, and let $A$ be the corresponding index set, that is $A = \{e \mid \Phi_e \in \mathcal{A}\}$. By Rice's Theorem, $A$ is an undecidable set (unless $A$ is trivial). These considerations show that we should take care when study decidability in computable analysis. If we ask stupid questions, the answer to each and on of them will be given by a straightforward corollary of Rice's Theorem. Next to nothing will be decidable.

Let us study a natural decidability problem. We have a realizer $\alpha$. Will $\alpha$ realize the computable real number 0, that is, do we have $\{\alpha\} = 0$? Intuitively, this is an undecidable problem. We can compute $\alpha(n)$ for bigger and bigger $n$ and check if $0 \in \underline{\alpha(n)}$. If we encounter an $n$ such that $0 \notin \underline{\alpha(n)}$, then we can conclude that $\alpha$ will not realize 0, but if we never encounter such an $n$, what happens then? Let us say that for every $n$ we have checked so far we have $\alpha(n) = 0$, or maybe $\alpha(n) = 2^{-n}$

(and thus $0 \in \underline{\alpha(n)}$). We cannot stop and conclude that $\{\alpha\} = 0$. It might very well be the case that we have $0 \notin \underline{\alpha(n)}$ for some big $n$ that we have not yet checked out. So intuitively, given a realizer $\alpha$, it is undecidable if $\{\alpha\} = 0$. Let us discuss how we can prove this insight.

Our considerations above shows that we should not prove that the set

$$A \;=\; \{\; e \mid e \text{ is an index for a realizer } \alpha \text{ and } \{\alpha\} = 0 \;\}$$

is undecidable. It follows trivially Rice's theorem that $A$ is undecidable, and moreover, that $A$ is undecidable does not reflect intuitive insight we want to prove. The insight we want to prove is given by the claim below.

> **Claim (ii)** Let $e$ be a computable index for a realizer. There is no computable function $h$ such that
>
> $$h(e) = \begin{cases} 0 & \text{if } \{\Phi_e\} = 0 \\ 1 & \text{otherwise.} \end{cases} \tag{2.17}$$

Note that the claim does not specify how $h$ should behave on input $e$ if $e$ is not in an index for a realizer. If $e$ is not in index for a realizer, then anything might happen: the computation of $h(e)$ might, or might not, terminate; the output might, or might not, be 0 or 117 or $10^{182} - 7$ or ....

The proof of Claim (ii) is similar to the proof of Claim (i). Let

$$\xi(x, n) = \begin{cases} 2^{-t} & \text{where } t \text{ is the least } t \text{ such that } t < n \text{ and } \mathcal{T}_1(x, x, t)] \text{ holds} \\ 0 & \text{if such a } t \text{ does not exist.} \end{cases}$$

We observe that $\xi$ is a total computable function ($\xi$ is primitive recursive). Let $f(x) = S_1^1(e, x)$ where $e$ is a computable index for $\xi$. Then $\Phi_{f(x)}$ is a realizer for a computable number strictly greater then 0 if $x \in \mathcal{K}$, and a realizer for 0 if $x \notin \mathcal{K}$. Thus, if a function $h$ satisfying (2.17) were computable, we could decide membership in $\mathcal{K}$ by computing $h(f(x))$. This proves Claim (ii).

> **Claim (iii)** Let $e_1, e_2$ be a computable indexes for realizers. There is no computable function $g$ such that
>
> $$g(e_1, e_2) = \begin{cases} 0 & \text{if } \{\Phi_{e_1}\} = \{\Phi_{e_2}\} \\ 1 & \text{otherwise.} \end{cases} \tag{2.18}$$

We prove Claim (iii). Assume we have a computable function $g$ satisfying (2.18). Let $\alpha$ be a realizer given by $\alpha(n) = 0$ (for all $n$)). Fix some computable index $e_0$ for $\alpha$. Let $h(x) = g(x, e_0)$. The $h$ is computable function satisfying (2.17). This contradicts Claim (ii) which says that no computable function has such a property. This proves Claim (iii).

Recall Theorem 2.20 which states that one, and only one, of

$$\{\alpha\} = \{\beta\} \qquad \{\alpha\} < \{\beta\} \qquad \{\beta\} < \{\alpha\} \ .$$

holds when $\alpha$ and $\beta$ are realizers. The next claim states that we cannot efficiently decide which on of the three alternatives that is fulfilled.

> **Claim (iv)** Let $e_1, e_2$ be a computable indexes for realizers. There is no computable function $f$ such that
>
> $$f(e_1, e_2) = \begin{cases} 0 & \text{if } \{\Phi_{e_1}\} < \{\Phi_{e_2}\} \\ 1 & \text{if } \{\Phi_{e_1}\} = \{\Phi_{e_2}\} \\ 2 & \text{if } \{\Phi_{e_1}\} > \{\Phi_{e_2}\} \ . \end{cases} \tag{2.19}$$

We prove the claim. Assume we have a computable function $f$ satisfying (2.19). Let

$$g(e_1, e_2) = \begin{cases} 0 & \text{if } f(e_1, e_2) = 1 \\ 1 & \text{if } f(e_1, e_2) = 0 \vee f(e_1, e_2) = 0 \ . \end{cases}$$

Then $g$ is a computable function satisfying (2.18). This contradicts Claim (iii).

Thus, even if we can prove that exactly one of the tree alternatives given by Theorem 2.20 holds, we cannot decide which one. This reflects that our proof of the theorem is not constructive. However, if we know (assume) that the alternative $\{\alpha\} = \{\beta\}$ does not hold, then we can decide which one of the two other that holds. That is what the next claim says.

> **Claim (v)** Let $e_1, e_2$ be a computable indexes for realizers such that $\{\Phi_{e_1}\} \neq \{\Phi_{e_2}\}$ . There exists a computable function $f$ such that
>
> $$f(e_1, e_2) = \begin{cases} 0 & \text{if } \{\Phi_{e_1}\} < \{\Phi_{e_2}\} \\ 1 & \text{if } \{\Phi_{e_1}\} > \{\Phi_{e_2}\} \ . \end{cases} \tag{2.20}$$

In order to see that Claim (v) holds, let

$$g(e_1, e_2) \;=\; \mu n\big[\; \underline{\{\Phi_{e_1}\}(n)}_r < \underline{\{\Phi_{e_2}\}(n)}_\ell \;\;\vee\;\; \underline{\{\Phi_{e_2}\}(n)}_r < \underline{\{\Phi_{e_1}\}(n)}_\ell \;\big] \,. \qquad (2.21)$$

The $g$ is computable function, moreover, the unbounded search will alway terminate when $\{\Phi_{e_1}\} \neq \{\Phi_{e_2}\}$. Let

$$f(e_1, e_2) = \begin{cases} 0 & \text{if } \underline{\{\Phi_{e_1}\}(g(e_1, e_2))}_r < \underline{\{\Phi_{e_2}\}(g(e_1, e_2))}_\ell \\ 1 & \text{otherwise .} \end{cases}$$

Then $f$ is a computable function which satisfies (2.20), and thus Claim (v) holds.

In (2.21) we resort to unbounded search. Intuitively this is necessary. Intuitively, the will be no computable bound for the $\mu$-operator. Let us prove that this indeed is the case, or at least, let us prove that the next claim holds.

> **Claim (vi)** Let $e_1, e_2$ be a computable indexes for realizers such that $\{\Phi_{e_1}\} \neq \{\Phi_{e_2}\}$ . There is no primitive recursive function $f$ such that
>
> $$f(e_1, e_2) = \begin{cases} 0 & \text{if } \{\Phi_{e_1}\} < \{\Phi_{e_2}\} \\ 1 & \text{if } \{\Phi_{e_1}\} > \{\Phi_{e_2}\} \,. \end{cases} \qquad (2.22)$$

In order to prove the claim, we need a set that is computable, but not primitive recursive. Such sets exists (se exercise in Section 3.6). Let $A$ be such a set and let $\chi_A$ be the characteristic function of the set. Furthermore, let $e_A$ be a computable index for $\chi_A$, and let

$$g(x, n) = \begin{cases} 2^{-t} & \text{if } (\exists t < n)[\; \mathcal{T}_1(e_A, x, t) \;\wedge\; \mathcal{U}(t) = 0 \;] \\ -(2^{-t}) & \text{if } (\exists t < n)[\; \mathcal{T}_1(e_A, x, t) \;\wedge\; \mathcal{U}(t) = 1 \;] \\ 0 & \text{otherwise.} \end{cases}$$

Now, in order to understand what is going on, fix $x$ (such that $g(x, n)$ becomes a function of $n$ alone) and observe that

- if $x \in A$, then $g(x, n)$ is a realizer for a number slightly bigger than 0
- if $x \notin A$, then $g(x, n)$ is a realizer for a number slightly smaller than 0.

Now, $g$ is a computable function, and thus by the S-m-n Theorem, we have a primitive recursive function $\xi$ such that $\Phi_{\xi(x)}(n) = g(n)$ (let $\xi(x) = S^1_1(e_g, x)$ where $e_g$ is a computable index for $g$) , and thus

- $\{\Phi_{\xi(x)}\} \neq 0$
- $\{\Phi_{\xi(x)}\} > 0$ if $x \in A$
- $\{\Phi_{\xi(x)}\} < 0$ if $x \notin A$.

Let $e_0$ be a computable index for a realizer of the computable real number 0, and assume for the sake of a contradiction that there is a primitive recursive function $f$ such that (2.22) holds. We have $\chi_A(x) = f(e_0, \xi(x))$, and thus, $\chi_A$ is a primitive recursive function, and thus, $A$ is a primitive recursive set, contrary to our assumption. This prove Claim (vi).

## 2.7 Exercises

Let $\alpha$ and $\beta$ be the realizers given by $\alpha(n) = 2^{-n}$ and $\beta(n) = 3^{-n}$ (for all $n \in \mathbb{N}$). Furthermore, let $\gamma$ be the realizer given by $\gamma(n) = n/(n+1)$ (for all $n \geq 1$).

**Exercise 1 (solution)**

Prove that $\{\alpha\} = \{\beta\}$.

**Exercise 2 (solution)**

Describe (informally) the number realized by $\{\alpha\}$ and $\{\beta\}$.

**Exercise 3 (solution)**

Prove that $\{\alpha\} < \{\gamma\}$.

**Exercise 4 (solution)**

Find a realizer $\delta$ such that $\{\alpha\} < \{\delta\}$ and $\{\delta\} < \{\gamma\}$.

**Exercise 5**

Prove that the (unbounded) $\mu$-operator in the definition of $(\alpha + \beta)$ can be replaced by the bounded $\mu$-operator. Try to provide a fairly sharp bound.

**Exercise 6**

Prove that the (unbounded) $\mu$-operator in the definition of $(\alpha \times \beta)$ can be replaced by the bounded $\mu$-operator. Try to provide a fairly sharp bound.

**Exercise 7**

Explain why the unbounded $\mu$-operator in the definition of $(\alpha \dot{-} \beta)$ cannot be replaced by the bounded $\mu$-operator.

**Exercise 8 (Solution)**

We define the computable function $\alpha : \mathbb{N} \to \mathbb{Q}$ recursively by $\alpha(0) = 4$ and

$$\alpha(s+1) \;=\; \begin{cases} \alpha(s) - \frac{4}{2(s+1)+1} & \text{if } s \text{ is even.} \\ \\ \alpha(s) + \frac{4}{2(s+1)+1} & \text{if } s \text{ is odd} \end{cases}$$

We have e.g.

$$\alpha(1) = 4 - \frac{4}{3}$$

and

$$\alpha(5) = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} \; .$$

Prove that $\alpha$ is a realizer.

**Exercise 9 (Solution)**

Let $\alpha$ be the realizer given in the previous exercise. Describe (very informally) the computable real number $\{\alpha\}$.

**Exercise 10**

Let $a, b$ be computable real numbers such that $a < b$. Prove that there exists a computable real number $c$ such that $a < c < b$.

**Exercise 11 (Solution)**

In order to prove that the structure $(\mathbb{R}_c, \hat{0}, \hat{1}, \hat{+}, \hat{\times})$ satisfies $(\forall x)[x+0=x]$, we need to prove that we have $(\alpha + 0^*) \equiv \alpha$ for any realizer $\alpha$. See (the proof of) Theorem 2.16. Prove carefully that $(\alpha + 0^*) \equiv \alpha$ holds for any realizer $\alpha$.

**Exercise 12**

Complete the proof of Theorem 2.21, that is, prove that the 11th axiom in Figure 2.2 is satisfied.

**Exercise 13 (Solution)**

Let $\alpha$ and $\beta$ be realizers. Prove that the following two assertions are equivalent:

1. $\underline{\alpha(n)}$ and $\underline{\beta(m)}$ overlap for any $n, m \in \mathbb{N}$
2. $\underline{\alpha(n)}$ and $\underline{\beta(n)}$ overlap for any $n \in \mathbb{N}$.

# Interludium: Classical Analysis and the Axiom of Continuity

One of our main concerns will be to compare computable analysis to classical analysis.

All the theorems of classical analysis follows from a collection of axioms. These are the axioms of an ordered field, see Figure 2.2. In addition to the axioms of an ordered field, which all are of first-order, we have one more axiom: *the axiom of continuity*. Let us see how Kuratowski [7] formulated this axiom in his textbook "Introduction to Calculus" from 1946:

> *... if we divide the set of all real numbers into two sets $A$ and $B$ such that any number belonging to the set $A$ is less than any number belonging to set $B$, then there are two possibilities: either there exists a greatest number of the set $A$ or there exists a least number of the set $B$ (it is being assumed that both sets are non-empty).*

The axiom of continuity is a second-order axiom (Kuratowski calls the axiom the *the principle of continuity*). When we formulate Kuratowski's statement in a formal language, we have to quantify over sets of real numbers and write something like

$$
\begin{aligned}
(\forall A)(\forall B)\{\ [\ (\exists x)[x \in A]\ \ \wedge\ \ (\exists x)[x \in B]\ \wedge\ & \\
(\forall x)[x \in A \vee x \in B]\ \ \wedge\ \ (\forall x)(\forall y)[x \in A \wedge y \in B \rightarrow x < y]\ ] & \\
\rightarrow & \\
(\exists z)[\ (z \in A \wedge (\forall x)[x \in A \rightarrow x \leq z])\ \vee & \\
(z \in B \wedge (\forall x)[x \in B \rightarrow z \leq x])\ ]\ \} &
\end{aligned}
$$

The axioms of classical analysis has only model up to isomorphism.

**Example 2.26.** The rational numbers ($\mathbb{Q}$) with the standard ordering ($<$) and standard addition ($+$) and multiplication ($\times$) satisfies all the axioms of classical analysis except the axiom of continuity. Let $A = \{x \in \mathbb{Q} \mid x \times x \leq 2\}$ and let $B = \{x \in \mathbb{Q} \mid x \times x > 2\}$. Then, $A$ and $B$ are nonempty sets and any number in $A$ is less than any number in $B$. Moreover, any rational number will either be in $A$ or $B$. Still, since $\sqrt{2}$ is not a rational, there is neither a greatest number in $A$ nor a least number in $B$. □

The preceding example shows that if a theorem holds for the real numbers, but not for the rational numbers, then we need the axiom of continuity to prove the theorem. We will prove a few theorems of classical analysis below, and thus, we will need the axiom. It will be convenient to us to use a slightly different formulation than the one given by Kuratowski above. We will use the formulation that you are likely to find in a modern textbook.

**Definition 2.27.** *A real number $a$ is an* upper bound *of a set of real numbers $A$ if $A \neq \emptyset$ and we have $x \leq a$ for any $x \in A$.*

*We say that* a set of real numbers is bounded above *if there exists an upper bound of the set (note that the empty set is not bounded above).*

*A set of real numbers $A$ has a* least upper bound *if there exists a real number $a$ such that*

- *$a$ is an upper bound of $A$, and*
- *if $b$ is an upper bound of $A$, then we have $a \leq b$.*

*We say that $a$ is the* least upper bound *of $A$.*

**Axiom 2.28** (Axiom of Continuity)**.** *If a set of real numbers is bounded above, then it has a least upper bound.*

**Example 2.29.** Let $A = \{x \in \mathbb{R} \mid x \times x \leq 2\}$. We have $a < 17$ for any $a \in A$. Thus, 17 is an upper bound of $A$. Moreover, $A$ is not empty. Thus, by the Axiom of Continuity, $A$ has a least upper bound. It is easy to see that $\sqrt{2}$ is the least upper bound of $A$. Note that $\sqrt{2}$ is also the least upper bound of the set $\{x \in \mathbb{R} \mid x \times x < 2\}$. □

More on classical analysis, and in particular, on the development and history of classical analysis, can be found in Boman & Rogers [3].

# Chapter 3

# Sequences of Reals

## 3.1 Basics

**Definition 3.1.** *Let $a_0, a_1, a_2, \ldots$ be a sequence of computable real numbers. We say that $a_0, a_1, a_2, \ldots$ is a* computable sequence *if there is a computable function $\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ such that*

- *the function $\alpha_n$ is a realizer where $\alpha_n$ is given by $\alpha_n(x) = \phi(n, x)$*
- *$a_n = \{\alpha_n\}$ for all $n \in \mathbb{N}$.*

*We say that $\phi$ is a* realizer *of a sequence and that $\phi$* realizes *the sequence $a_0, a_1, a_2, \ldots$.*

Next we recall a well know definition from classical analysis.

**Definition 3.2** (Classical)**.** *A real number $b$ is* the limit *of the sequence $a_0, a_1, a_2, \ldots$ of real numbers if there for any $\epsilon > 0$ exists $k$ such that*

$$n \geq k \quad \Rightarrow \quad |a_n - b| < \epsilon \, .$$

*We say that a sequence of real numbers* converges*, or* is convergent*, if there exists a limit of the sequence.*

The definition above tells us that an element $a_i$ of a converging sequence will be arbitrarily close to the limit $b$ when $i$ is bigger than some $k$, but the definitions yields no information about this $k$ except that it is there, and the definition yields

no information about the elements preceding the $k$th element of the sequence, e.g., the finite initial segement $a_0, a_1, \ldots, a_{k-1}$ may display a natural pattern indicating that the sequence will converge to something very different from $b$. Definition 3.2 does not make sense in computable analysis, but we can find a definition that make sense and still capture our intuitive notion of a limit:

**Definition 3.3** (Computable)**.** *A computable real number $b$ is* the limit *of the computable sequence $a_0, a_1, a_2, \ldots$ if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that*

$$n \geq f(k) \quad \Rightarrow \quad |a_n - b| < (k+1)^{-1} \, .$$

*We say that a computable sequence* converges*, or* is convergent*, if there exists a limit of the sequence.*

In general, for any definition in classical analysis, we can try to find a corresponding definition in computable analysis, that is, a definition which (more or less) capture the same intuitive notion and which makes sense in computable setting. We have the classical definition of a Cauchy sequence:

**Definition 3.4** (Classical)**.** *A sequence $a_0, a_1, a_2, \ldots$ of real numbers is a* Cauchy sequence *if there for any positive real number $\epsilon$ exists $k$ such that*

$$m, n \geq k \quad \Rightarrow \quad |a_m - a_n| < \epsilon \, .$$

When we have seen how the limit of a sequence is defined in computable analysis, it is not hard see how we should Cauchy sequence.

**Definition 3.5** (Computable)**.** *A computable sequence $a_0, a_1, a_2, \ldots$ is a* Cauchy sequence *if there exist a computable function $f : \mathbb{N} \to \mathbb{N}$ such that*

$$m, n \geq f(k) \quad \Rightarrow \quad |a_m - a_n| < (k+1)^{-1} \, .$$

Since a classical definition may have a computable substitute (or maybe several computable substitues), a classical theorem may also have a computable substitute. Theorem 3.7 below is a computable version (analog, substitute, replacement, ...) of Theorem 3.6.

**Theorem 3.6** (Classical)**.** *A sequence of real numbers converges if and only if it is a Cauchy sequence.*

**Theorem 3.7** (Computable)**.** *A computable sequence of real numbers converges if and only if it is a Cauchy sequence.*

Does the computable version of theorem hold? Yes, it does, an we will give a proof below. But there are plenty of classical theorems where computable analogs do not hold. We will look at a few of them.

We will use $\lim_n a_n$ to denote the limit of the convergent sequence $a_0, a_1, a_2, \ldots$ (when we use the notation it is understood that the limit exists). It is not very surprising that converging sequences and limits in computable analysis obeys laws like

- $\lim_n(a_n + b_n) = \lim_n a_n + \lim_n b_n$
- $\lim_n(a_n \times b_n) = \lim_n a_n \times \lim_n b_n$
- $\lim_n(a_n - b_n) = \lim_n a_n - \lim_n b_n$
- if $\lim_n b_n \neq 0$, then $\lim_n(a_n \div b_n) = \lim_n a_n \div \lim_n b_n$.

## 3.2 The Proof of Theorem 3.6 and Theorem 3.7

The two proofs given in this section should be compared to each other. We prove the theorem of computable analysis first.

### 3.2.1 The Proof of Theorem 3.7

First we prove that any convergent sequence is a Cauchy sequence. Assume that the computable sequence $a_0, a_1, a_2, \ldots$ converges to $b$ (we will prove that $a_0, a_1, a_2, \ldots$ is a Cauchy sequence according to Definition 3.5). By Definition 3.3, we have a computable function $f$ such that

$$n \geq f(k) \quad \Rightarrow \quad |a_n - b| < (k+1)^{-1} .$$

Let $g(x) = f(2x+1)$. Obviously, $g$ is a computable functions such that

$$n \geq g(k) \quad \Rightarrow \quad |a_n - b| < (2k+2)^{-1} . \tag{3.1}$$

We will prove that we also have

$$m, n \geq g(k) \;\; \Rightarrow \;\; |a_m - a_n| < (k+1)^{-1} \tag{3.2}$$

and thus, by Definition 3.5, $a_0, a_1, a_2, \ldots$ is a Cauchy sequence.

In order to prove (3.2), assume $m, n \geq g(k)$. Then, we have

$$\begin{aligned} |a_m - a_n| &= |a_m - b + b - a_n| \\ &\leq |a_m - b| + |b - a_n| && \text{(by Th. 2.25 )} \\ &< (2k+2)^{-1} + (2k+2)^{-1} && \text{by (3.1)} \\ &= (k+1)^{-1} \, . \end{aligned}$$

This proves that (3.2) holds, and thus, $a_0, a_1, a_2, \ldots$ is a Cauchy sequence.

Next, we prove that any Cauchy sequence $a_0, a_1, a_2, \ldots$ is convergent. Assume that $a_0, a_1, a_2, \ldots$ is a Cauchy sequence. By Definition 3.5, we have a computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$m, n \geq f(k) \;\; \Rightarrow \;\; |a_m - a_n| < (k+1)^{-1} \tag{3.3}$$

We can without loss of generality assume that $f(x) \leq f(x+1)$. Let $\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a realizer of the sequence $a_0, a_1, a_2, \ldots$. Then we have that $\alpha_n$ is a realiser of $a_n$ where $\alpha_n(x) = \phi(n, x)$. We define the computable function $\beta$ by

$$\beta(n) = \phi(f(2n+1), 2n+1) \, .$$

We will prove that

$$\beta \text{ is a realizer} \tag{claim I}$$

and that

$$\beta \text{ is the limit of } a_0, a_1, a_2, \ldots \; . \tag{claim II}$$

It follows from these two claims that $a_0, a_1, a_2, \ldots$ is convergent. Before we proceed and prove our claims, we will state and discuss a rather trivial lemma.

**Lemma 3.8.** *Let $\gamma$ be a realizer of $c \in \mathbb{R}_c$. Then, for any $n \in \mathbb{N}$*

$$|c - \gamma(n)| \; \leq \; (n+1)^{-1} \, .$$

Be aware that we abuse notation in this lemma (in the same way as we have abused notation several places above, e.g., when we defined convergent computable sequences). We claim something of the form $|c - p| \leq q$ where $p, q \in \mathbb{Q}$ and $c \in \mathbb{R}_c$, and strictly speaking, this does not make sense. What we actually claim is that we have $|c - \{p^*\}| \leq \{q^*\}$ where $p^*$ and $q^*$ are the realizers given by $p^*(n) = p$ and $q^*(n) = q$ (for all $n \in \mathbb{N}$). The statement $|c - \{p^*\}| \leq \{q^*\}$ make sense as we have $c, \{p^*\}, \{q^*\} \in \mathbb{R}_c$, see Section 2.3 page 22. It easy that the lemma holds when read the lemma as it should be read, and we leave the proof as an exercise for the reader.

Let us turn to the proof of (caim I). By Definition 2.1 we have to prove that

$$|\beta(n) - \beta(m)| \ \leq \ (n+1)^{-1} + (m+1)^{-1} \, . \tag{3.4}$$

holds for any $n, m \in \mathbb{N}$. By the definition of $\beta$ and Lemma 3.8, we have

$$\begin{aligned} |\beta(n) - a_{f(2n+1)}| \ = \ |\phi(f(2n+1), 2n+1) - a_{f(2n+1)}| \ = \\ |\alpha_{f(2n+1)}(2n+1) - a_{f(2n+1)}| \ \leq \ (2n+2)^{-1} \, . \end{aligned} \tag{3.5}$$

Let $k = \min(n, m)$. We have $f(2n+1), f(2m+1) \geq f(2k+1)$, and by (3.3), we have

$$|a_{f(2n+1)} - a_{f(2m+1)}| \ < \ (2k+2)^{-1} \ = \ (2\min(n,m)+2)^{-1} \, . \tag{3.6}$$

By the definition of $\beta$ and Lemma 3.8, we have

$$\begin{aligned} |a_{f(2m+1)} - \beta(m)| \ = \ |a_{f(2m+1)} - \phi(f(2m+1), 2m+1)| \ = \\ |a_{f(2m+1)} - \alpha_{f(2m+1)}(2m+1)| \ \leq \ (2m+2)^{-1} \, . \end{aligned} \tag{3.7}$$

Thus, we have

$$\begin{aligned} |\beta(n) - \beta(m)| \quad &= \quad |\beta(n) - a_{f(2n+1)} + a_{f(2n+1)} - a_{f(2m+1)} + a_{f(2m+1)} - \beta(m)| \\ &\quad \text{(by Th. 2.25 )} \\ &\leq \quad |\beta(n) - a_{f(2n+1)}| + |a_{f(2n+1)} - a_{f(2m+1)}| + |a_{f(2m+1)} - \beta(m)| \\ &\quad \text{(by (3.5), (3.6) and (3.7))} \\ &< \quad (2n+2)^{-1} + (2\min(n,m)+2)^{-1} + (2m+2)^{-1} \\ &< \quad (n+1)^{-1} + (m+1)^{-1} \, . \end{aligned}$$

This proves that (3.4) holds, and thus we conclude that $\beta$ is a realizer.

We have seen that $\beta$ is a realizer, and thus, $\{\beta\}$ will be a computable real. We will now prove our second claim, namely that $\{\beta\}$ is the limit of the sequence $a_0, a_1, a_2, \ldots$. First me make a couple of observations. By Lemma 3.8, we have

$$|\beta(4k+3) - \{\beta\}| \;\leq\; (4k+4)^{-1} \tag{3.8}$$

for any $k \in \mathbb{N}$. Moreover, by the definition of $\beta$ and Lemma 3.8, we have

$$|a_{f(4k+4)} - \beta(4k+3)| \;=\; |a_{f(4k+4)} - \phi(f(4k+4), 4k+4)| \;= \\ |a_{f(4k+4)} - \alpha_{f(4k+4)}(4k+4)| \;\leq\; (4k+5)^{-1} \tag{3.9}$$

for any $k \in \mathbb{N}$.

We are now ready to prove

$$n \geq f(2k+1) \;\Rightarrow\; |a_n - \{\beta\}| < (k+1)^{-1} \qquad \text{(claim III)}$$

It is not hard to see that (claim II) will follow easily from (claim III).

In order to prove (claim III), assume $n \geq f(2k+1)$. We obviously have $f(4k+4) \geq f(2k+1)$, and by (3.3), we get

$$|a_n - a_{4k+4}| \;<\; (2k+2)^{-1} \tag{3.10}$$

and thus we have

$$\begin{aligned}
|a_n - \{\beta\}| \quad &= \quad |a_n - a_{f(4k+4)} + a_{f(4k+4)} - \beta(4k+3) + \beta(4k+3) - \{b\}| \\
&\qquad \text{(by Th. 2.25 )} \\
&\leq \quad |a_n - a_{f(4k+4)}| + |a_{f(4k+4)} - \beta(4k+3)| + |\beta(4k+3) - \{b\}| \\
&\qquad \text{(by (3.10), (3.9) and (3.8))} \\
&< \quad (2k+2)^{-1} + (4k+5)^{-1} + (4k+4)^{-1} \\
&< \quad (2k+2)^{-1} + (4k+4)^{-1} + (4k+4)^{-1} \\
&= \quad (k+1)^{-1} \,.
\end{aligned}$$

This proves that (claim III) holds. Let $g(x) = f(2x+1)$. Then $g$ is a computable function, and by (claim III), we have

$$n \geq g(k) \;\Rightarrow\; |a_n - \{\beta\}| < (k+1)^{-1} \,.$$

Hence, by Definition 3.3, $a_0, a_1, a_2, \ldots$ is a convergent sequence of computable real numbers. This proves (claim II).

### 3.2.2 The Proof of Theorem 3.6

First we prove that any convergent sequence is a Cauchy sequence. Let $b$ be the limit of $a_0, a_1, a_2, \ldots$. Let $\epsilon > 0$ be given. By Definition 3.2, we have $k$ such that

$$\ell \geq k \;\; \Rightarrow \;\; |a_\ell - b| < \epsilon/2 \; .$$

Assume $m, n \geq k$. Then, we have

$$|a_n - a_m| \;=\; |a_n - b + b - a_m| \;\leq\; |a_n - b| + |b - a_m| \;<\; \epsilon/2 + \epsilon/2 \;=\; \epsilon \; .$$

This proves that $a_0, a_1, a_2, \ldots$ is a Cauchy sequence.

Next we prove that any Cauchy sequences converges. Our proof uses the Bolzano-Weierstrass Theorem. A sequence $a_0, a_1, a_2, \ldots$ is bounded if there exists $M$ such that $-M < a_n < M$ (for alll $n$).

**Theorem 3.9** (Bolzano-Weierstrass)**.** *Any bounded sequence contains a convergent subsequence.*

A proof of the Bolzano-Weierstrass Theorem requires the Axiom of Continuity, and you will find a proof of the theorem in many standard introductory texts (including Kuratowski [7] and Boman & Rogers [3]).

Let $a_0, a_1, a_2, \ldots$ be a Cauchy sequence (we will prove that $a_0, a_1, a_2, \ldots$ is convergent). Let $\epsilon > 0$ be given.

By Definition 3.4, we have $k_0$ such that

$$m, n \geq k_0 \;\; \Rightarrow \;\; |a_m - a_n| < \epsilon/3 \; . \tag{3.11}$$

It is easy to see that we have $M$ such that $-M < a_n < M$ (for all $n$). Use (3.11) with 3 for $\epsilon$. Thus, by the Bolzano-Weierstrass Theorem we have a convergent subsequence $a_{\ell_0}, a_{\ell_1}, a_{\ell_2}, \ldots$. Let $b = \lim_i a_{\ell_i}$. Note that we have $\ell_m \geq m$ (for any $m$), and by Definition 3.2, we have $k_1$ such that

$$m \geq k_1 \;\; \Rightarrow \;\; |a_{\ell_m} - b| < \epsilon/3 \tag{3.12}$$

and moreover, by (3.11), we have

$$m, n \geq k_0 \;\; \Rightarrow \;\; |a_{\ell_n} - a_m| < \epsilon/3 \; . \tag{3.13}$$

Let $n \geq \max(k_0, k_1)$. Then, pick some $m \geq k_0$, by (3.11), (3.13) and (3.12), we have

$$\begin{aligned} |a_n - b| \; = \; |a_n - a_m + a_m - a_{\ell_n} + a_{\ell_n} - b| \; \leq \; |a_n - a_m| + |a_m - a_{\ell_n}| + a_{\ell_n} - b| \\ \leq \; \epsilon/3 + \epsilon/3 + \epsilon/3 \; = \; \epsilon \; . \end{aligned}$$

This proves that $a_0, a_1, a_2, \ldots$ is convergent.

## 3.3 Numerical Expansions of Reals

We are used to represent real numbers by numerical expansions in base 10. We may write

$$\pi \; = \; 3.141592 \ldots \; .$$

This is a way to state that

$$\pi \;\; = \;\; 3 + \frac{1}{10^1} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} \cdots \; .$$

Every $a \in \mathbb{R}$ can be written of the form

$$a \;\; = \;\; M + \sum_{i=1}^{\infty} \frac{\mathtt{D}_i}{10^i}$$

where $M \in \mathbb{Z}$ and $\mathtt{D}_i \in \{0, \ldots, 9\}$.

The base-10 expansion of a real number is unique. Oooops... that was not correct! We have eg. $\frac{1}{2} = 0.5000000 \ldots$ and $\frac{1}{2} = 0.499999999 \ldots$. Some rational numbers have two base-10 expansions, but every irrational number has a unique base-10 expansion. Some rational numbers also have a unique base-10 expansion, e.g., the one and only base-10 expansion of $\frac{1}{3}$ is $0.333333333 \ldots$.

In daily life we write up numbers in base 10. This is to some extent an arbitrary convention. In principle we can expand real numbers in any base $b$ where $b$ is a natural number greater than or equal to 2, but in practice we cannot have a convention saying that we should write up numbers in, e.g., base 317. It would be an insurmountable task to learn and remember all the digits. Neither would it

be a good idea to use base 2 in daily life. Even relatively small numbers become hard to read and communicate if we write them in base 2. People who have to work with numbers in base 2 (bits: 0,1) do often convert these number to base 16 (hexadecimals: 0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F). It is easy to convert on the fly between base-2 and base-16, and numbers in base-16 are far easier to communicate. (It is not all that easy to convert on the fly between base 2 and base 10.)

However, expansions of reals in base-2 is what we need to complete some of our proof. We will restrict our attention to reals lying strictly between 0 and 1. Every real $a$ in the interval $(0,1)$ can be uniquely written of the form

$$a \;=\; 0 + \sum_{i=1}^{\infty} \frac{\mathtt{D}_i}{2^i} \;=\; \sum_{i=0}^{\infty} \frac{\mathtt{D}_i}{2^i} \tag{3.14}$$

where $\mathtt{D}_i \in \{0,1\}$ and

$$\sum_{i=1}^{n} \frac{\mathtt{D}_i}{2^i} \;\leq\; a \;<\; \sum_{i=0}^{n} \frac{\mathtt{D}_i}{2^i} + \frac{1}{2^n} \tag{3.15}$$

for all $n \geq 1$. The strict inequalty in (3.15) ensures that a rational number does not have two base-2 expansions. We will us $E_2^a(i)$ to denote the function that yields the $i^{\text{th}}$ digit in the binary expansion of $a \in (0,1)$. The formal definition of the function follows.

**Definition 3.10.** *Let $a$ be a real number in the interval $(0,1)$. We define $E_2^a : \mathbb{N} \to \{0,1\}$ as the unique function such that we have*

$$\sum_{i=0}^{n} \frac{E_2^a(i)}{2^i} \;\leq\; a \;<\; \sum_{i=0}^{n} \frac{E_2^a(i)}{2^i} + \frac{1}{2^n}$$

*for any $n \in \mathbb{N}$.*

**Lemma 3.11.** *Let $a$ be a computable real number in the interval $(0,1)$. Then $E_2^a$ is a computable function.*

*Proof.* The proof splits into the case (i) $a$ is rational and (ii) $a$ is irrational. If $a$ is rational, then it is obvious that $E_2^a$ is a computable function.

Assume that $a$ is irrational, and let $a = \{\alpha\}$ where $\alpha$ is a realizer. Assume by induction hypothesis that we can compute $E_2^a(i)$ when $i < n$. We show how to compute $E_2^a(n)$.

- (Step 1) Compute $q := \sum_{i=0}^{n-1} \frac{E_2^a(i)}{2^i} + \frac{1}{2^n}$.
- (Step 2) Search for $s$ such that $\underline{\alpha(s)}_r < q$ or $q < \underline{\alpha(s)}_\ell$.
- (Step 3) Let $E_2^a(n) = 0$ if there exists $s$ such that $\underline{\alpha(s)}_r < q$; otherwise, let $E_2^a(n) = 1$.

Unbounded search is required in (Step 2), but the search will terminate since $\{\alpha\}$ is an irrational number. □

## 3.4 Specker Sequences

The notion of a bounded sequence makes sense in both classical and computable analysis. We say that a (computable) sequence $a_0, a_1, a_2, \ldots$ of reals is *bounded* if there exists $b \in \mathbb{R}$ ($b \in \mathbb{R}_c$) such that $a_i < b$ for all $i \in \mathbb{N}$.

**Theorem 3.12** (Classical)**.** *Every monotone increasing and bounded sequence of reals converges.*

*Proof.* Let $a_0, a_1, a_2, \ldots$ be a bounded sequences. The sequences is monotone increasing, that is, $a_i \leq a_{i+1}$ for all $i$. Let $A = \{a_i \mid i \in \mathbb{N}\}$. The set $A$ is of course bounded, and by the axiom of continuity $A$ has a least upper bound $b$. By our definition, a sequence converges if it has a limit. We will prove that $b$ is the limit of the sequence $a_0, a_1, a_2, \ldots$.

By the definition of a limit (Definition 3.2), we have to prove that there for any positive real number $\epsilon$ exists $k$ such that

$$n \geq k \quad \Rightarrow \quad |a_n - b| < \epsilon \,. \tag{3.16}$$

So, pick an arbitrary real number $\epsilon > 0$. If we have $a_n \leq b - \epsilon$ for all $n$, the $b$ cannot be the least upper bound of $A$. Hence, we have $b - \epsilon < a_k$ for some sufficiently large $k$. As the sequence is increasing, we have $b - \epsilon < a_k \leq a_n$ for all $n \geq k$. Hence we also have $b - a_n < \epsilon$ for all $n \geq k$. This proves (3.16). □

The next theorem shows that the computable analog of Theorem 3.12 does not hold. The theorem was first proved by Specker [13], and the sequence $q_0, q_1, q_2, \ldots$ constructed in the proof of the theorem is called a *Specker sequence*.

**Theorem 3.13** (Specker)**.** *There exists a strictly monotone increasing and bounded computable sequence of computable reals that does not converge.*

*Proof.* Let $A$ be a set that is computably enumerable, but not computable ($A$ may e.g. be the set $\mathcal{K}$, see Definition 1.8). There is a total computable one-to-one function $f_A$ such that $A = \mathrm{rng}(f_A)$. We define the sequence $q_0, q_1, q_2, \ldots$ of rationals by $q_0 = 0$ and

$$q_{i+1} \;=\; q_i + \frac{1}{2^{f_A(i)+1}}\,.$$

There is a computable function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ is given by $f(n,m) = q_n^*(m)$ (for all $m, n \in \mathbb{N}$, see Definition 2.15). Let $\alpha_n(m) = f(n,m)$. Then $\alpha_n$ is a realizer of the rational number $q_n$. Hence, $f$ realizes a computable sequence $a_0, a_1, a_2, \ldots$. The sequence is bounded as any number in the sequence is less than 1. The sequences is strictly monotone increasing as we obviously have $q_i < q_{i+1}$.

**(claim)** *For any $b \in \mathbb{R}_c$ there exists $k \in \mathbb{N}$ such that $|a_n - b| > \frac{1}{k}$ for all but finitely many values of $n$.*

Our theorem follows straightforwardly from the claim above and Definition 3.3.

We prove the claim. The claim obviously holds if $b \leq 0$ or $1 \leq b$. Assume that $0 < b < 1$. By Lemma 3.11, $E_2^b$ is a computable function. The function $\chi$ where

$$\chi(x) \;=\; \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x - 1 \in A \text{ and } x > 0 \\ 0 & \text{if } x - 1 \notin A \text{ and } x > 0. \end{cases}$$

is not computable (if $\chi$ were a computable function, $A$ would be a computable set). Hence, we cannot have $E_2^b(x) = \chi(x)$ for all $x$. Let $n$ be a least $n$ such that $E_2^b(n) \neq \chi(n)$. Thus we have $n > 0$, as $E_2^b(0) = \chi(0) = 0$, and

$$\sum_{i=0}^{n-1} \frac{E_2^b(i)}{2^i} \;=\; \sum_{i=0}^{n-1} \frac{\chi(i)}{2^i}\,. \tag{3.17}$$

Moreover, since $E_2^b$ and $\chi$ are 0-1 valued functions, it is either the case that (i) $E_2^b(n) = 0$ and $\chi(n) = 1$, or it is the case that (ii) $E_2^b(n) = 1$ and $\chi(n) = 0$. We will prove that the claim holds in both cases.

*Case (i): $E_2^b(n) = 0$ and $\chi(n) = 1$.* We will need

**(*)** for any $n$ there exists $j$ such that $\displaystyle\sum_{i=0}^{n} \frac{\chi(i)}{2^i} \;<\; q_j$.

We leave the straightforward proof of (*) to the reader. By (*), we have $j$ such that

$$\begin{array}{lll}
\displaystyle\sum_{i=0}^{n} \frac{E_2^b(i)}{2^i} & \leq & b \\
& < & \displaystyle\sum_{i=0}^{n} \frac{E_2^b(i)}{2^i} + \frac{1}{2^n} \qquad \text{(by definition of } E_2^b\text{)} \\
& = & \displaystyle\sum_{i=0}^{n} \frac{\chi(i)}{2^i} \qquad \text{((3.17), } E_2^b(n) = 0 \text{ and } \chi(n) = 1\text{)} \\
& < & q_j \;<\; q_{j+1} \;<\; \ldots \qquad \text{(*)}
\end{array}$$

Let $k$ be a natural number such that $1/k < (q_{j+1} - q_j)$ and (claim) holds.

*Case (ii):* $E_2^b(n) = 1$ *and* $\chi(n) = 0$. We have

$$q_j \;<\; \sum_{i=0}^{n} \frac{E_2^b(i)}{2^i}$$

for all $j$. If $b > \sum_{i=0}^{n} \frac{E_2^b(i)}{2^i}$, we have $m$ such that

$$q_j \;<\; \sum_{i=0}^{n} \frac{E_2^b(i)}{2^i} \;<\; \sum_{i=0}^{n} \frac{E_2^b(i)}{2^i} + \frac{1}{2^m} \;\leq\; b$$

for all $j$. Let $k = 2^m$ and the (claim) holds.

Assume $b = \sum_{i=0}^{n} \frac{E_2^b(i)}{2^i}$. Let $m$ be such that $m > n$ and $\chi(m) = 0$. Such an $m$ exists, otherwise $A$ would be a computable set. Let $k = 2^m$ and (claim) holds. □

## 3.5 Sequences and Countability

A nonempty set $A$ is *countable* if there exists a surjection $f : \mathbb{N} \to A$ (this is a pretty standard definition). Thus, the sequence $f(0), f(1), f(2), \ldots$ will contain all the elmenets of $A$. Note that it is not required that the surjection $f$ should be computable in any sense of the word. The definition just says that such a surjection should exist. The set $\mathbb{R}$ of classical real numbers is known to be uncountable, and that is what the next theorem says.

**Theorem 3.14** (Cantor, classical)**.** *No sequence of reals contains all the reals.*

*Proof.* Let $a_0, a_1, a_2, \ldots$ be any sequence of reals. We will prove that there is a real number $b$ such that $b \neq a_i$ (for all $i \in \mathbb{N}$).

First we construct the sequence $b_0, b_1, b_2, \ldots$. Let $b_0 = 2^{-1}$ and let

$$b_{i+1} \;=\; \begin{cases} b_i + 2^{-2i} & \text{if } a_i \leq b_i \\ b_i - 2^{-2i} & \text{if } a_i > b_i. \end{cases}$$

We have $|b_{i+1} - b_i| \leq 2^{-2i}$. Thus, the sequence is a Cauchy sequence, and by Theorem 3.6, the sequence converges. Let $b = \lim_i b_i$.

It is not very hard to see that

$$b_i < b_{i+1} \;\Rightarrow\; b_i < b \tag{3.18}$$

and

$$b_i > b_{i+1} \;\Rightarrow\; b_i > b \tag{3.19}$$

hold for all $i \in \mathbb{N}$ (observe that $2^{-2i} - \sum_{j=i+1}^{\infty} 2^{-2j} \;>\; 2^{-2i-1}$).

Assume for the sake of a contradiction that $b$ occurs in the sequence $a_0, a_1, a_2, \ldots$, that is, assume there exists $m$ such that $a_m = b$. The proof splits into the two cases $a_m \leq b_m$ and $a_m > b_m$. In both cases we will encounter a contradiction.

Let $a_m \leq b_m$. Then

$$b \;=\; a_m \;\leq\; b_m \;<\; b_m + 2^{-2m} \;=\; b_{m+1}$$

and by (3.18), we have $b = a_m \leq b_m < b$. This is a contradiction.

Let $a_m > b_m$. Then

$$b \;=\; a_m \;>\; b_m \;>\; b_m - 2^{-2m} \;=\; b_{m+1}$$

and by (3.19), we have $b = a_m > b_m > b$ which is a contradiction. □

The set $\mathbb{R}_c$ of computable real numbers is countable. That is a consequence of the Enumeration Theorem (Theorem 1.6). Let $f(e) = \Phi_e$. Then $f$ is a surjection from $\mathbb{N}$ into the set of unary computable functions. Since each realizer is a unary

computable function, and each computable real number is realized by a realizer, there will exist a surjection $g : \mathbb{N} \to \mathbb{R}_c$, and the sequence $g(0), g(1), g(2), \ldots$ will contains all the computable real numbers. There must exists such a $g$, but be aware: we are claiming anything beyond mere existence, and in particular, we are not claiming that $g(0), g(1), g(2), \ldots$ will or might be a computable sequence. Indeed, our next the theorem, which is a computable analog of Theorem 3.14, states that no computable sequence contains all computable reals. Before we turn to the proof of that theorem we will do some preliminary work.

We will work with interval with rational endpoints. An intervall $I$ is an ordered pair of rationals $(p, q)$ such that $p, q$. It does not matter if work with open or closed intervals, and let us say that a number $x$ is in the intervall $I = (p, q)$ if $p < x < q$. For any intervall $I = (p, q)$ we define

$$L(I) = (p, (q - p)/3) \quad \text{and} \quad R(I) = (q - (q - p)/3, q)$$

and thus $L(I)$ denotes the first one third of the intervall $I$ wheres $R(I)$ denotes the last one third. The next lemma should not surprise anyone.

**Lemma 3.15.** *Let $\alpha$ be an realizer and let $I$ be an intervall. For all sufficiently large $n$, the intervall $\underline{\alpha(n)}$ will not overlap with both $L(I)$ and $R(I)$ (and thus $\underline{\alpha(n)}$ will overlap with at most one of them).*

*Proof.* Let $I = (p, q)$. Pick any $n$ such that $2/(n + 1) < R(I)_\ell - L(I)_r = (q - p)/3$. The length of the intervall $\underline{\alpha(n)}$ is $2/(n + 1)$. The distance between the intervalls $L(I)$ and $R(I)$ is $(q - p)/3$ Thus $\underline{\alpha(n)}$ cannot overlap with with both $L(I)$ and $R(I)$. □

**Theorem 3.16** (Cantor, computable)**.** *No computable sequence of reals contains all the computable reals.*

*Proof.* Let $a_0, a_1, a_2, \ldots$ be any computable sequence of reals. Let $\phi$ be a realiser for the sequence. Let $\alpha_n(x) = \phi(n, x)$. Then, $\alpha_n$ is a realizer for $a_n$.

We will now construct a sequence $I_0, I_1, I_2, \ldots$ of intervals such that $a_n \notin I_{n+1}$ and $I_{n+1} \subset I_n$.

Let $I_0 = (0, 1)$. In order to determine $I_{n+1}$ we search for the least $m$ such that $\underline{\alpha_n(m)}$ does not overlap with both with both $L(I)$ and $R(I)$. This search terminates

by Lemma 3.15. Then we let

$$I_{n+1} \quad = \quad \begin{cases} L(I_n) & \text{if } \underline{\alpha_n(m)} \text{ overlaps with } R(I_n) \\ R(I_n) & \text{otherwise.} \end{cases}$$

Let $\beta(n)$ be the midpoint of the intervall $I_n = (p_n, q_n)$, that is, $\beta(n) = (q_n - p_n)/2$. We claim that $\beta$ is a realizer for a computable real number that does not occur in the sequence $a_0, a_1, a_2, \ldots$.

It is fairly obvious that $\beta$ is a computable function. We have prove that $\beta$ is a realizer, that is, we have to prove that $\underline{\beta(n)}$ and $\underline{\beta(m)}$ overlap for any $n, m$.

Observe that we have $I_n \subseteq \underline{\beta(n)}$ for all $n$ (the two intervals have the same midpoint, and $I_n$ is a much shorter interval than $\underline{\beta(n)}$, length $3^{-n}$ versus length $2(n+1)^{-1}$). Furthermore, we have $I_{n+1} \subseteq I_n$ by our construction of the intervalls. Hence, let $q$ be any number in the intervall $I_{\max(m,n)}$, then we have $q \in \underline{\beta(n)}$ and $q \in \underline{\beta(m)}$. This proves that $\beta$ is a realizer, and thus, $\{\beta\}$ is a computable real number.

In order to prove that $\{\beta\} \neq a_n$ (for all $n \in \mathbb{N}$), we will provide $k$ and $m$ such that $\underline{\beta(k)}$ and $\underline{\alpha_n(m)}$ do not overlap. By our construction of the interval $I_n$, there exists $m$ such that $\underline{\alpha_n(m)}$ does not overlap with $I_n = (p_n, q_n)$. We can w.l.o.g. assume that $\underline{\alpha_n(m)}$ lies at the right hand side of $I_n$, that is, we can w.l.o.g. assume $q_n < \underline{\alpha_n(m)}_\ell$ (a symmetric argument will take care of the case when $\underline{\alpha_n(m)}_r < p_n$). Pick $k$ such that $k \geq n$ and $(k+1)^{-1} < \underline{\alpha_n(m)}_\ell - q_n$. Then $\underline{\beta(k)}$ cannot overlap with $\underline{\alpha_n(m)}$ since $\beta(k) \in I_k \subseteq I_n$ and the distance from $\beta(k)$ to the right endpoint $\underline{\beta(k)}_r$ is $(k+1)^{-1}$. This proves that the computable real number $\{\beta\}$ does not occur in the computable sequence $a_0, a_1, a_2, \ldots$. □

Theorem 3.14 and Theorem 3.16 are proved by diagonalization. This is a proof technique which goes back to Georg Cantor. The reader should compare the two proofs. The proof of Theorem 3.14 is a straightforward application of the technique. The proof Theorem 3.16 is also an application of the techinque, and the two proofs are in some sense very similar, but in the latter proof the details are not all that straightforward, and some effort is needed to make the diagonalization argumet work.

**Definition 3.17.** *A computable real number is* primitive recursive *if it has a primitive recursive realizer.*

The set $\mathbb{R}$ of real numbers is uncountable. The set $\mathbb{R}_c$ of computable real numbers is countable, but due to Theorem 3.16 it makes pretty good sense to say that set set is *computably uncountable*. If we stick to this terminology, the set of primitive recursive real numbers will be *computably countable*.

**Theorem 3.18.** *There exists a computable sequence of real numbers such that (i) every number in the sequence is primitive recursive and (ii) every primitive recursive real number occur in the sequence.*

*Proof.* Let $\mathrm{PR}(e)$ be the predicate that holds if and only if $e$ is a primitive recursive index, that is, $e$ is a computable index for a function generated from the initial functions (zero, successor, projections) by the scheme of composition and the scheme of primitive recursion (see Definition 1.1 and Definition 1.3). It is obvious that $\mathrm{PR}(e)$ is a computable predicate, and it is obvious that we for any primitve recursive $f : \mathbb{N} \to \mathbb{N}$ have $f(x) = \Phi_e(x)$ for some $e$ such that $\mathrm{PR}(e)$ holds.

Any primitive recursive function $f : \mathbb{N} \to \mathbb{N}$ can be considered as a total computable function $f : \mathbb{N} \to \mathbb{Q}$ if we are careful when we set up our coding conventions (just ensure that any natural number encode some rational number). Thus, for any primitive recursive $f$ and any $x \in \mathbb{N}$, the intervall $\underline{f(x)}$ is well defined.

Let $\mathrm{OK}(e, n)$ be the predicate that holds if and only if $\mathrm{PR}(e)$ holds, and moreover, $\underline{\Phi_e(x)}$ and $\underline{\Phi_e(y)}$ overlap for any $x, y \leq n$. Given the comments above, it should be fairly obvious that $\mathrm{OK}(e, n)$ is a computable (decidable) predicate, and next we use this predicate to define a computable function $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

If $\mathrm{PR}(e)$ does not hold, let $\psi(e, x) = (17)^{-1}$ for all $x$ (any rational number will work, so why not chose $(17)^{-1}$, it is an attractive number). If $\mathrm{PR}(e)$ holds, we define $\psi(e, x)$ by

$$\psi(e,x) \;=\; \begin{cases} \Phi_e(x) & \text{if } \mathrm{OK}(e,x) \\ \kappa_e & \text{otherwise.} \end{cases}$$

with

$$\kappa_e \;=\; \frac{1}{2}\left( \left( \max_{i<k} \underline{\Phi_e(i)}_\ell \right) \;+\; \left( \min_{i<k} \underline{\Phi_e(i)}_r \right) \right)$$

where $k$ is the least natural number such that $\mathrm{OK}(e, k)$ does not hold.

> (Claim) Let $\psi_e(x) = \psi(e, x)$. For any $e \in \mathbb{N}$, there exists a primitive recursive realizer $\alpha$ such that $\alpha \equiv \psi_e$.

In order to prove the claim we pick an arbitrary natural number $e$. If $e$ is not a primitive recursive index, that is, if $\mathrm{PR}(e)$ does not hold, we have $\psi_e(x) = (17)^{-1}$ for all $x$. Let $\alpha(x) = (17)^{-1}$ for all $x$, and the claim holds. If $\mathrm{PR}(e)$ holds we have to deal with two cases: The case when $\mathrm{OK}(e, x)$ holds for all $x$, and the case when there exists $k$ such that $\mathrm{OK}(e, k)$ does not hold.

Assume $\mathrm{OK}(e, x)$ holds for all $x$. Then we have a primitive recursive index $e$ such that $\psi_e(x) = \Phi_e(x)$ (for all $x$). Thus, there is a primitive recursive function $\alpha$ such that $\alpha(x) = \psi_e(x)$. Obviously, $\alpha$ is a realizer, and obviously, $\alpha \equiv \psi_e$. Assume there exists $k$ such $\mathrm{OK}(e, k)$ does not hold. Let $k'$ be the least such $k$. The intervalls

$$\underline{\Phi_e(0)}\ ,\ \underline{\Phi_e(1)}\ ,\ \ldots\ ,\ \underline{\Phi_e(k'-1)}$$

overlap, and the rational number $\kappa_e$ will be in them all. Thus, by the definition of $\psi_e$ we have $\kappa_e \in \underline{\psi_e(x)}$ for all $x$. This proves that $\psi_e$ is a realizer. Let $\alpha(x) = \kappa_e$ (for all $x$). Then $\alpha$ is a primitive recursive realizer, moreover, $\underline{\alpha(x)}$ and $\underline{\psi_e(y)}$ overlap for any $x$ and $y$, and thus $\alpha \equiv \psi_e$. This concludes the proof of the claim.

It follows from the claim that $\psi$ realizes a sequence and that any number in this sequence is a primitive recursive real, and thus clause (i) of our theorem holds. It is easy to see that clause (ii) holds: Let $\alpha$ be a primitive recursive realizer. Then we have $\alpha(x) = \Phi_e(x)$ for some $e$ such that $\mathrm{PR}(e)$ holds. By the definiton of $\psi$ we have $\alpha(x) = \psi(e, x)$, and thus the number realized by $\alpha$ will occur in the sequence realized by $\psi$. □

**Corollary 3.19.** *There exist computable real numbers that are not primitive recursive.*

*Proof.* By Theorem 3.18 there is a computable sequence of reals $a_0, a_1, a_2, \ldots$ that contain all the primitive recursive reals. By Theorem 3.16 there exists a computable real number $b$ that is not in the sequence $a_0, a_1, a_2, \ldots$. Thus, $b$ cannot be primitive recursive. □

## 3.6 Exercises

**Exercise 1.**

Prove that Axiom 2.28 is equivalent to the axiom formulated by Kuratowski in the quote at page 36.

**Exercise 2.**

Give a reasonable formalization of Axiom 2.28. Use propositional connectives, first-order quantifiers, second-order quantifiers and the relation symbols $\leq$ and $\in$.

**Exercise 3**

Discuss why Definition 3.2 and Definition 3.4 do not work in computable setting.

**Exercise 4 (solution)**

Let $\alpha$ and $\alpha'$ be any realizers such that $\{\alpha\} < \{\alpha'\}$. Show that there exists a computable sequence $a_0, a_1, a_2, \ldots$ of computable reals such that

$$\{\alpha\} < a_0 < a_1 < \ldots < a_i < a_{i+1} < \ldots < \{\alpha'\}\,.$$

Use the definition of *computable sequence* given above (Definition 3.1).

**Exercise 5**

Let $a$ be a real number in the interval $(0, 1)$. Prove that $a$ is a *computable* real number if $E_2^a$ is computable function.

**Exercise 6**

Prove that there exists a computable function $\xi$ such that $\xi(e)$ is a computable index for a realizer of $a$ if $e$ is computable index for $E_2^a$. (Informally stated: we can compute a realizer of $a$ uniformly in the base-2 expansion of $a$.)

**Exercise 7**

Let $\psi$ be a function such that $\psi(e)$ is a computable index for $E_2^a$ if $e$ is a computable index for realizer of $a$. Prove that $\psi$ is not computable. (Informally stated: we cannot compute the base-2 expansion of $a$ uniformly in an realizer of $a$.) This is a hard exercise, see Mostowski [9].

**Exercise 8**

Let $b, c$ be any computable reals such that $b < c$. Prove that there exists a Specker sequence between $b$ and $c$.

**Exercise 9 (solution)**

Prove that there exists a computable (decidable) set that is not primitive recursive. Use Corollary 3.19 and Lemma 3.11.

**Exercise 10 (solution)**

Use Corollary 3.19 to prove that there exists a computable and total function $f : \mathbb{N} \to \mathbb{N}$ that is not bounded by any primitive recursive function, that is, for any primitive recursive $g$ it is not the case that $g(x) \geq f(x)$ holds for all $x$.

**Exercise 11 (solution)**

Corollary 3.19 claims the existence of computable real numbers that are not primitive recursive. The argument in the proof shows that there is at least one such number, but there will of course be infinitely many of them. Explain why the set of such numbers is dense in the rational numbers, that is, let $(p, q)$ be an interval with rational endpoints, explain why there is a computable real number in $(p, q)$ that is not primitive recursive.

# Chapter 4

# Computable Functions

## 4.1 Basics

We will use $\widehat{\phi}$ to denote an arbitrary computable index for the computable function $\phi$. Every time we use the notation $\widehat{\phi}$ we have implicitly stated in advance that $\widehat{\phi}$ is a computable index for $\phi$ (recall that each computable function has infinitly many indices).

**Definition 4.1.** *Let $\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a computable function such that for any realizers $\alpha, \alpha_1, \alpha_2$*

*(A) $\beta$ is a realizer if $\beta(n) = \phi(\widehat{\alpha}, n)$*

*(B) $\beta_1 \equiv \beta_2$ if $\beta_1(n) = \phi(\widehat{\alpha_1}, n)$, $\beta_2(n) = \phi(\widehat{\alpha_2}, n)$ and $\alpha_1 \equiv \alpha_2$.*

*For any realizer $\alpha$, let $f(\{\alpha\}) = \{\beta\}$ where $\beta(n) = \phi(\widehat{\alpha}, n)$. Then $f$ is a function from $\mathbb{R}_c$ into $\mathbb{R}_c$. We say that $f$ is the function* given *by $\phi$ (and that $\phi$* gives *$f$). A function $f : \mathbb{R}_c \to \mathbb{R}_c$ is* computable *if it is given by some computable function.*

**Theorem 4.2.** *Any constant function is computable. The identity function is computable.*

*Proof.* Let $f : \mathbb{R}_c \to \mathbb{R}_c$ be such that $f(x) = a$ for all $x \in \mathbb{R}_c$. Let $\alpha$ be any realizer of $a$, and let $\phi(y, n) = \alpha(n)$. Then, $f$ is given by $\phi$. Thus, by Definition 4.1, $f$ is a computable function.

Let $g : \mathbb{R}_c \to \mathbb{R}_c$ be such that $g(x) = x$ for all $x \in \mathbb{R}$. Let

$$\psi(y, n) \;=\; \mathcal{U}(\; \mu t[\mathcal{T}_1(y, n, t)] \;)$$

where the predicate $\mathcal{T}_1$ and the function $\mathcal{U}$ are given by Kleene's Normal Form Theorem. Then, $g$ is given by the computable function $\psi$, and thus, $g$ is computable by Definition 4.1. □

**Theorem 4.3.** *Let $|\cdot|$, $+$, $\times$, $-$, $\div$ be the operators defined in Section 2.1. Then*

1. *$|f(x)|$ is a computable function if $f$ is a computable function*
2. *$f(x) + g(x)$ is a computable function if $f$ and $g$ are computable functions*
3. *$f(x) \times g(x)$ is a computable function if $f$ and $g$ are computable functions*
4. *$f(x) - g(x)$ is a computable function if $f$ and $g$ are computable functions*
5. *$f(x) \div g(x)$ is a computable function if $f$ and $g$ are computable functions and $g(x) \neq 0$ for all $x$ (we will normally use $\frac{f(x)}{g(x)}$ to denote the function $f(x) \div g(x)$).*

*Proof.* We prove clause (2). The proofs of the other clauses are similar.

Let $f : \mathbb{R}_c \to \mathbb{R}_c$ be the computable function given by $\phi$, and let $g : \mathbb{R}_c \to \mathbb{R}_c$ be the computable function given by $\psi$. Let

$$\xi(x, n) = \phi(x, f_+(x, n)) + \psi(x, f_+(x, n))$$

where

$$\begin{aligned} f_+(n) \;=\; (\mu i) \Bigg[ & \left| \; \phi(x,i) + \psi(x,i) - (\underline{\phi(x,i)}_\ell + \underline{\psi(x,i)}_\ell) \; \right| \;\leq\; \frac{1}{n+1} \quad \wedge \\ & \left| \; \phi(x,i) + \psi(x,i) - (\underline{\phi(x,i)}_\ell + \underline{\psi(x,i)}_r) \; \right| \;\leq\; \frac{1}{n+1} \quad \wedge \\ & \left| \; \phi(x,i) + \psi(x,i) - (\underline{\phi(x,i)}_r + \underline{\psi(x,i)}_\ell) \; \right| \;\leq\; \frac{1}{n+1} \quad \wedge \\ & \left| \; \phi(x,i) + \psi(x,i) - (\underline{\phi(x,i)}_r + \underline{\psi(x,i)}_r) \; \right| \;\leq\; \frac{1}{n+1} \Bigg] \; . \end{aligned}$$

We need to prove that the computable function $\xi$ satisfies (A) and (B) of Definition 4.1, that is, we need to prove that we for any realizers $\alpha, \alpha_1, \alpha_2$ have

(A) $\beta$ is a realizer if $\beta(n) = \xi(\widehat{\alpha}, n)$

(B) $\beta_1 \equiv \beta_2$ if $\beta_1(n) = \xi(\widehat{\alpha_1}, n)$, $\beta_2(n) = \xi(\widehat{\alpha_2}, n)$ and $\alpha_1 \equiv \alpha_2$.

The proof of (A) is very similar to the proof of Theorem 2.9, the proof of (B) is very similar to the proof of Theorem 2.10, and thus we leave the details to the reader. For any realizer $\alpha$, let $h(\{\alpha\}) = \{\beta\}$ where $\beta(n) = \xi(\widehat{\alpha}, n)$. Definition 4.1, $h : \mathbb{R}_c \to \mathbb{R}_c$ is a computable function.

Let $\alpha$ be an arbitrary realizer. Let

- $\beta$ be the realizer given by $\beta(n) = \xi(\widehat{\alpha}, n)$.
- $\alpha_1$ be the realizer given by $\alpha_1(n) = \phi(\widehat{\alpha}, n)$.
- $\alpha_2$ be the realizer given by $\alpha_2(n) = \psi(\widehat{\alpha}, n)$.

Recall the definition of $(\alpha_1 + \alpha_2)$, that is, Definition 2.7, and look at the definition of $\xi$ above. It should be straightforward to prove that we have $\beta \equiv (\alpha_1 + \alpha_2)$, and thus we have $h(x) = f(x) + g(x)$. □

**Theorem 4.4.** *The function $g(f(x))$ is computable if $f$ and $g$ are computable.*

*Proof.* Let $f$ be given by $\phi$, and let $g$ be given by $\psi$. Let $\xi(x, n) = \psi(S_1^1(\widehat{\phi}, x), n)$ where $S_1^1$ is the primitive recursive function given by the S-m-n Theorem. We prove that $g(f(x))$ is given by the computable function $\xi$.

Let $\alpha$ be a realizer, and let $\beta(n) = \phi(\widehat{\alpha}, n)$. Then $\beta$ is a realizer of $f(\{\alpha\})$. Furthermore, we have

$$\beta(n) = \phi(\widehat{\alpha}, n) = \{\widehat{\phi}\}(\widehat{\alpha}, n) = \{S_1^1(\widehat{\phi}, \widehat{\alpha})\}(n)$$

and thus, $S_1^1(\widehat{\phi}, \widehat{\alpha})$ is a computable index for $\beta$.

Let $\gamma(n) = \psi(S_1^1(\widehat{\phi}, \widehat{\alpha}), n)$. Then, $\gamma$ is a realizer of $g(\{\beta\})$ as $S_1^1(\widehat{\phi}, \widehat{\alpha})$ is a computable index for $\beta$, and $g(\{\beta\}) = g(f(\{\alpha\}))$. Hence, let $\xi(x, n) = \psi(S_1^1(\widehat{\phi}, x), n)$, and $g(f(x))$ is given by $\xi$. □

**Definition 4.5.** *Let $a, b \in \mathbb{R}_c$. We define the operators* max *and min by*

$$\max(a, b) \quad = \quad \begin{cases} a & \text{if } a \geq b \\ b & \text{if } b > a \end{cases}$$

*and*

$$\min(a,b) \;=\; \begin{cases} b & \text{if } a \geq b \\ a & \text{if } b > a. \end{cases}$$

**Theorem 4.6.** *Let $f$ and $g$ be computable functions. Then* $\max(f(x), g(x))$ *and* $\min(f(x), g(x))$ *are computable functions.*

*Proof.* We have

$$\max(f(x), g(x)) = \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}\,|f(x) - g(x)|$$

and

$$\min(f(x), g(x)) = \frac{1}{2}(f(x) + g(x)) - \frac{1}{2}\,|f(x) - g(x)|\ .$$

Thus, the theorem follows the theorems above. □

It follows from the theorems above that functions like $x^2 + \sqrt{2}x + 17$ and $\frac{4}{3}\pi x^3$ are computable ($\sqrt{2}$ and $\pi$ are computable real numbers). It also follows that the function

$$\max\left(\ \frac{4^{-1} - |x - 2^{-1}|}{4^{-1}}\ ,\ 0\ \right) \tag{4.1}$$

is computable (its graph is shown in Figure 4.1). Well known standard functions like $\sin(x)$, $\cos(x)$, $e^x$, and so on, will also be computable. However, our main interests are elsewhere. We want to construct a few computable functions that behave unexpectedly strange. Spike functions, like the one showed in Figure 4.1, will turn out to be useful in that respect.

## 4.2 A Little Bit Computability Theory

1.2

Recall the definitions of $\mathcal{K}$, $\mathcal{W}_x$, etc., found in Section 1.2.

**Definition 4.7.** *The function $\kappa : \mathbb{N} \to \mathbb{N}$ is given by*

$$\kappa(x) \;=\; \begin{cases} 1 \dot{-} \Phi_x(x) & \text{if } x \in \mathcal{K} \\ \textit{undefined} & \textit{otherwise.} \end{cases}$$

*A function $g : \mathbb{N}^n \to \mathbb{N}$ is an* extension *of a function $f : \mathbb{N}^n \to \mathbb{N}$ if $g(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$ whenever $f(x_1, \ldots, x_n)$ is defined.*

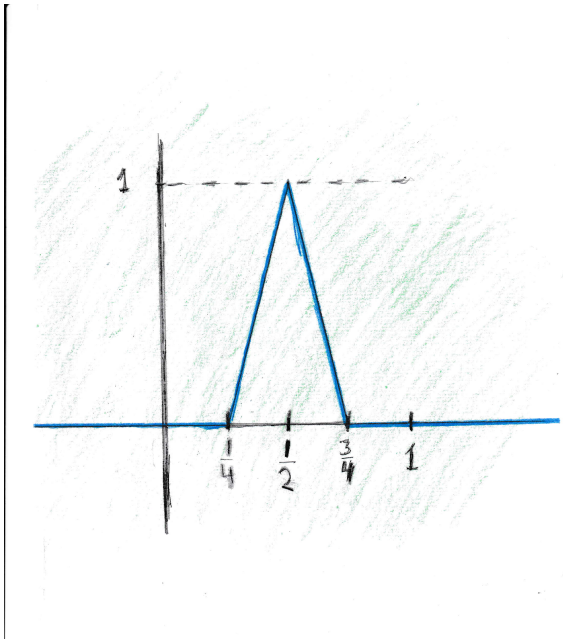



Figure 4.1: The graph of (4.1)

**Theorem 4.8.** *(i) $\kappa$ is a computable function. (ii) $\kappa$ has no total computable extension.*

*Proof.* First we argue that (i) holds. Let $\mathrm{rng}(f) = \mathcal{K}$. The following algorithm computes $\kappa(x)$:

- Step 1: search for $i$ such that $f(i) = x$ (if no such $i$ exists, then $\kappa(x)$ is undefined)
- Step 2: compute $a := \Phi_x(x)$
- Step 3: output $1 \dot{-} a$.

If the first step of the algorithm terminates, we have $x \in \mathcal{K}$. Thus, by Definition 1.8, we have $x \in \mathcal{W}_x = \mathrm{dom}(\Phi_x)$. Hence, the computation of $\Phi_x(x)$ in the second step will terminate. This proves (i).

Assume for the sake of a contradiction that $g$ is a total computable extension of $\kappa$. Let $e$ be computable index for $g$. Since $g$ is total we have $\mathcal{W}_e = \mathbb{N}$, and thus $e \in \mathcal{W}_e$.

Hence $e \in \mathcal{K}$. Hence,

$$\Phi_e(e) \;=\; g(e) \;=\; 1\dot{-}\Phi_e(e)\;.$$

This is a contradiction. □

**Definition 4.9.** *We will use $\mathbf{D}^2_i(q)$ to denote the $i^{th}$ fractional digit*[1] *of the binary expansion of the rational number $q$. More precisely, let $\lfloor q \rfloor$ denote the integral part of the rational $q$, then we have*

$$\sum_{i=1}^{n} \frac{\mathbf{D}^2_i(q)}{2^i} \;\leq\; q - \lfloor q \rfloor \;<\; \sum_{i=1}^{n} \frac{\mathbf{D}^2_i(q)}{2^i} + \frac{1}{2^n} \tag{4.2}$$

*for all $n \geq 1$.*

It is obvious that we can compute the digit $\mathbf{D}^2_i(q)$ when $i$ and $q$ are given. Observe that $0.0111111\ldots$ is not a possible binary expansion of $1/2$. The strict inequality in (4.2) assures that the one and only binary expansion of $1/2$ is $0.1000000\ldots$.

## 4.3 Some Weird Functions

We will use the function function $\kappa$, see Definition 4.7, to construct a Specker sequence $b_1, b_2, b_3, \ldots$ with certain desirable properties. Let $f : \mathbb{N} \to \mathbb{N}$ be a total computable injection such that $\mathrm{rng}(f) = \mathcal{K}$, and let

$$b_1 = 0 \quad \text{and} \quad b_{n+1} \;=\; b_n + \frac{1}{2^{4n}} \;+\; \frac{1}{2^{4f(n)+2}} \;+\; \frac{\kappa(f(n))}{2^{4f(n)+3}}\;.$$

We make the the following observations:

- (i) $b_n + 2^{-4n} < b_{n+1}$ (for all $n$)
- (ii) if $\mathbf{D}^2_{4i+2}(b_n) = 1$ and $m > n$, then $\mathbf{D}^2_{4i+3}(b_m) = \mathbf{D}^2_{4i+3}(b_n)$
- (iii) $\mathbf{D}^2_{4i}(b_{n+1}) = 1$ when $i \leq n$
- (iv) $\mathbf{D}^2_{4i+1}(b_n) = 0$ (for all $n, i$).

[1] the $i^{\text{th}}$ digit after the period

**(1st claim)** *Let $\alpha$ be any realizer. There exist $I$ and $N$ such that*

- $4I + 3 < N - 4$
- $\mathbf{D}^2_{4I+2}(b_N) = 1$ *and* $\mathbf{D}^2_{4I+3}(b_N) \neq \mathbf{D}^2_{4I+3}(\alpha(2^N))$.

Now, assume for the sake of contradiction that the claim is false. Let

$$\rho(x) \;=\; \mathbf{D}^2_{4x+3}(\alpha(2^{4(x+2)}))\;.$$

Then, $\rho$ is a total computable extension of $\kappa$. This contradicts Theorem 4.8. Thus the 1st claim holds.

We will keep the realizer $\alpha$ and the natural numbers $I$ and $N$ given by the 1st claim fixed in the foreseeable future.

**(2nd claim)** *For any $n \geq N$, we have*

$$\left|\sum_{i=1}^{N} \frac{\mathbf{D}^2_i(b_n)}{2^i} - \sum_{i=1}^{N} \frac{\mathbf{D}^2_i(\alpha(2^N))}{2^i}\right| \;>\; \frac{1}{2^{N-3}}\;.$$

In order to see that our second claim holds, observe that the first clam states that $\mathbf{D}^2_{4I+2}(b_N) = 1$ and $\mathbf{D}^2_{4I+3}(b_N) \neq \mathbf{D}^2_{4I+3}(\alpha(2^N))$. By observation (ii), we have $\mathbf{D}^2_{4I+3}(b_n) \neq \mathbf{D}^2_{4I+3}(\alpha(2^N))$ for any $n \geq N$. Thus, for any $n \geq N$, we also have

$$\left|\sum_{i=1}^{4I+3} \frac{\mathbf{D}^2_i(b_n)}{2^i} - \sum_{i=1}^{4I+3} \frac{\mathbf{D}^2_i(\alpha(2^N))}{2^i}\right| \;\geq\; \frac{1}{2^{4I+3}}\;.$$

Furthermore, by observation (iii) and (iv), we have $\mathbf{D}^2_{4I+4}(b_N) = 1$ and $\mathbf{D}^2_{4I+5}(b_N) = 0$. Thus, for any $n \geq N$, we have

$$\sum_{i=1}^{4I+3} \frac{\mathbf{D}^2_i(b_n)}{2^i} \;>\; \sum_{i=1}^{4I+3} \frac{\mathbf{D}^2_i(\alpha(2^N))}{2^i} \quad \Rightarrow$$
$$\left|\sum_{i=1}^{4I+5} \frac{\mathbf{D}^2_i(b_n)}{2^i} - \sum_{i=1}^{4I+5} \frac{\mathbf{D}^2_i(\alpha(2^N))}{2^i}\right| \;>\; \frac{1}{2^{4I+4}} \quad (4.3)$$

and

$$\sum_{i=1}^{4I+3} \frac{\mathbf{D}_i^2(\alpha(2^N))}{2^i} \;>\; \sum_{i=1}^{4I+3} \frac{\mathbf{D}_i^2(b_n)}{2^i} \quad \Rightarrow$$

$$\left| \sum_{i=1}^{4I+5} \frac{\mathbf{D}_i^2(b_n)}{2^i} - \sum_{i=1}^{4I+5} \frac{\mathbf{D}_i^2(\alpha(2^N))}{2^i} \right| \;\geq\; \frac{1}{2^{4I+4}} \,. \quad (4.4)$$

Now, $N \geq 4I + 8$, and thus, by (4.3) and (4.4), we have

$$\left| \sum_{i=1}^{N} \frac{\mathbf{D}_i^2(b_n)}{2^i} - \sum_{i=1}^{N} \frac{\mathbf{D}_i^2(\alpha(2^N))}{2^i} \right| \;\geq\; \frac{1}{2^{4I+4}} - \sum_{i=4I+6}^{N} \frac{1}{2^i} \;>\; \frac{1}{2^{4I+5}} \;\geq\; \frac{1}{2^{N-3}}$$

for any $n \geq N$. This proves our second claim.

It is not hard to see that we have

$$\left| \sum_{i=1}^{N} \frac{\mathbf{D}_i^2(\alpha(2^N))}{2^i} - \alpha(2^N) \right| \;\leq\; \frac{1}{2^N} \quad (4.5)$$

and

$$\left| \alpha(2^N) - \{\alpha\} \right| \;\leq\; \frac{1}{2^N} \,. \quad (4.6)$$

Neither is it hard to see that we have

$$\left| \sum_{i=1}^{N} \frac{\mathbf{D}_i^2(b_n)}{2^i} - b_n \right| \;\leq\; \frac{1}{2^N} \quad (4.7)$$

for all $n \geq N$. The inequalities (4.5), (4.6), (4.7) and the 2nd claim yield our next claim.

**(3rd claim)** *For any $n \geq N$, we have*

$$|b_n - \{\alpha\}| \;>\; \frac{1}{2^N} \,.$$

*Hence, the interval $\underline{\alpha(2^N)}$ cannot contain any $b_n$ in the sequence $b_1, b_2, b_3, \ldots$ when $n \geq N$.*

The last claim should be compared to the claim in the proof of Theorem 3.13 (page 48). It follows straightforwardly from (3rd claim) that $b_1, b_2, b_3, \ldots$ is a Specker

sequences, that is, $b_1, b_2, b_3, \ldots$ is a bounded and monotone increasing sequence that does not converge. But we have also proved something more: From (a computable index for) a realizer $\alpha$, we can indeed compute $N$ such that $b_n \notin \underline{\alpha(2^N)}$ if $n \geq N$. The algorithm is implicitly given by our first claim: search for $I$ and $N$ such that

- $1 \leq 4I + 3 < N - 4$
- $\mathbf{D}^2_{4I+2}(b_N) = 1$ and $\mathbf{D}^2_{4I+3}(b_N) \neq \mathbf{D}^2_{4I+3}(\alpha(2^N))$.

The claim guarantees that the search terminates.

We will now use the sequence $b_1, b_2, b_3, \ldots$ to define some interesting computable functions over $\mathbb{R}_c$. We start by defining a sequences of functions $t_2, t_3, \ldots$. For each $n \geq 2$, we define $t_n$ by

$$t_n(x) \;=\; \max\left( \frac{2^{-4n-1} - |x - b_n|}{2^{-4n-1}}, 0 \right) .$$

Then, for all $x \in \mathbb{R}_c$, we have $0 \leq t_n(x) \leq 1$, and $t_n$ is a spike function like the one shown in Figure 4.1. Moreover, by the theorems in Section 4.1, $t_n$ is a computable function. It also follows from the theorems in 4.1 that the function

$$t_2(x) + t_3(x) + \ldots + t_k(x)$$

is computable for any fixed $k$.

We define the function $t$ by

$$t(x) \;=\; \sum_{i=2}^{\infty} t_i(x) .$$

Let us try to visualize the graph of $t$. By observation (i) above, we have $b_{n+1} - b_n > 2^{-n}$, and thus, the spike of the function $t_n$ will not intersect the spike of the function $t_{n+1}$ (for all $n \geq 2$), see Figure 4.2. Since $0 < b_n < 1$, it is obvious $t(x)$ equals 0 outside the interval $[0, 1]$ (for all $n \geq 2$). On the the interval $[0, 1]$ there will be infinitely many spikes. Between two neighbor spikes there will be an interval $I$ where we have $t(x) = 0$ for any $x \in I$. As the spike of the function $t_{n+1}$ is sharper than the spike of the function $t_n$, the function $t$ will obviously not be uniformly continuous at the interval $[0, 1]$.

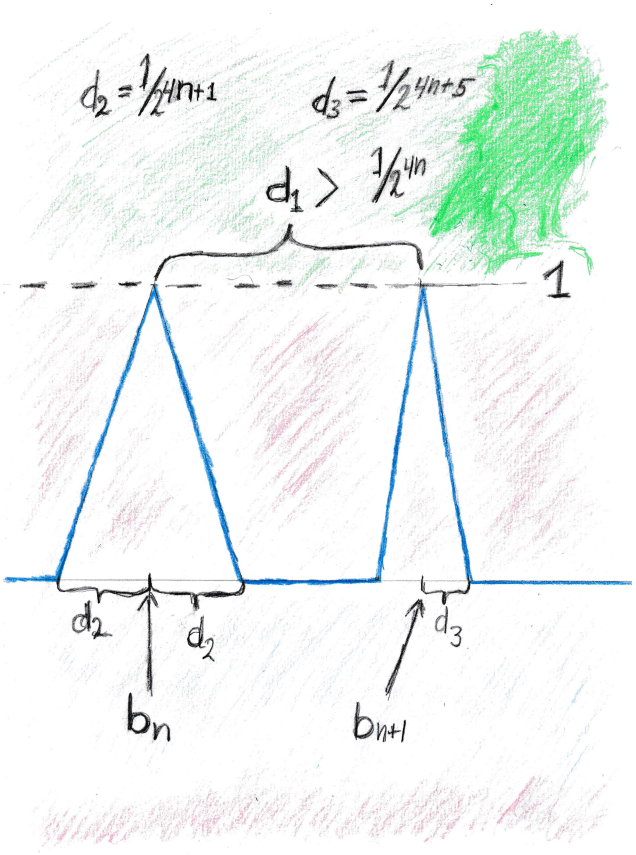



Figure 4.2: A bit of the graph of $t$

**(main claim)** *The function $t$ is computable.*

We prove the claim: From (an index for) a realizer $\alpha$, we can compute $N$ such that $b_n \notin \underline{\alpha(2^N)}$ if $n \geq N$ (we have argued above that we can compute such an $N$). Then we have

$$t(\{\alpha\}) \;=\; \sum_{i=2}^{\infty} t_i(\{\alpha\}) \;=\; \sum_{i=2}^{N} t_i(\{\alpha\})$$

and can compute $t(\{\alpha\})$ by computing $\sum_{i=2}^{N} t_i(\{\alpha\})$. The function $\sum_{i=2}^{N} t_i(x)$ is computable by the theorems in Section 4.1. This proves that $t$ is a computable function.

Let us also add that $t$ is a continuous function. Any computable function which is defined on an interval $I$, will be continious on $I$ (Ceitin [4], see Appendix 3 of Aberth [1]).

ABERTH: A definition *continuous* and *uniformly continuous* computable functions can be found in Section 7.1 of Aberth [1]. I might add those definitions to these notes at a later stage, but the definitions are not really needed to prove the next theorem. The function $t$ is not uniformly continuous in the classical sense, and thus it will not be uniformly continuous in the computable sense either. A proof that the function $t$ is continuous will require definitions in Aberth [1]. $\diamond$

The next theorem follows straightaway from the definition of $t$ and our main claim: (i), (ii) and (iii) follow from the definition of $t$, and the claim states that $t$ is computable.

**Theorem 4.10.** *There exists a computable (and thus continuous) function $t : \mathbb{R}_c \to \mathbb{R}_c$ such that (i) $t$ is defined on the interval $[0, 1]$, (ii) $0 \leq t(x) \leq 1$ and (iii) $t$ is not uniformly continuous on the interval $[0, 1]$.*

In classical analysis any function that is continues on a closed interval $I$ will of course also be uniformly continuous on $I$. Thus, classical version of Theorem 4.10 does not hold, neither do the classical versions of the two theorems below.

**Definition 4.11.** *Let $f : \mathbb{R}_c \to \mathbb{R}_c$ be a computable function defined on an interval $I$. An* upper bound *for $f$ on $I$ is a computable real number $m$ such that $f(x) \leq m$ for any $x \in I$. A* supremum *for $f$ on $I$ is a computable real number $m$ such*

- *$m$ is an* upper bound *for $f$ on $I$*

- *if $m$ and $m'$ are* upper bounds *for $f$ on $I$, then $m \leq m'$.*

**Theorem 4.12.** *There exists a computable (and thus continuous) function $t : \mathbb{R}_c \to \mathbb{R}_c$ such that (i) $t$ is defined on the interval $[0,1]$, (ii) $0 \leq t(x) \leq 1$ and (iii) $t$ does not have a supremum on the interval $[0,1]$.*

**Theorem 4.13.** *There exists a computable (and thus continuous) function $t : \mathbb{R}_c \to \mathbb{R}_c$ such that (i) $t$ is defined on the interval $[0,1]$ and (ii) $t$ does not have an upper bound on the interval $[0,1]$.*

The proofs of these two theorems are very similar to the proof of Theorem 4.10. In order to prove theorem 4.12, define $t$ by

$$t(x) \;=\; \sum_{i=2}^{\infty} b_i t_i(x) \; .$$

Then $t$ will not have a supremum since $b_1, b_2, b_3, \ldots$ is a Specker sequence. In order to prove theorem 4.12, define $t$ by

$$t(x) \;=\; \sum_{i=2}^{\infty} i t_i(x) \; .$$

## 4.4 More Details

We swept quite a few details under the rug when we proved our main claim in the previous section. We argued as follows:

- from an index for a realizer $\alpha$, we can compute $N$ such that $b_k \notin \underline{\alpha(2^N)}$ if $k \geq N$ *(1st premise)*
- $\sum_{i=1}^{N} t_i(x)$ is computable function, and we can compute $t(\{\alpha\})$ by computing $\sum_{i=1}^{N} t_i(\{\alpha\})$ *(2nd premise)*
- $t$ is computable *(conclusion)*.

According to Definition 4.1, $t : \mathbb{R}_c \to \mathbb{R}_c$ is computable if $t$ is given by some $\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. What does such a $\phi$ look like? How can we construct such a $\phi$? In our arguments above we implicitly left the construction of $\phi$ to the reader. Maybe we

expected too much from the reader, maybe not? If not, the reader might skip the remainder of this section.

Let us briefly discussed the theorems we proved in Section 4.1. We proved that $f(x)+g(x)$ is a computable fuctions if $f$ and $g$ are computable functions, we proved that $|f(x)|$ is a computable function if $f$ is a computable function, we proved that $g(f(n))$ is is a computable fuctions if $f$ and $g$ are computable functions, and so on. If we study the proofs, we will see that we actually have proved essentially more than what is explicitly stated in the theorems. The proofs tell us how we can compute indexes for the functions appearing in the theorems. E.g., one of the theorems states that

$f(x)+g(x)$ is a computable fuctions if $f$ and $g$ are computable functions.

The proof tells us that if

- $f(x)$ is given by $\psi_1$
- $g(x)$ is given by $\psi_2$

then we can uniformly in $\psi_1$ and $\psi_2$ construct $\psi_3$ such that

- $f(x)+g(x)$ is given by $\psi_3$.

The construction in the proof is efficient in the following sense: There exists a computable function $\eta$ such that $\eta(e_1,e_2)$ is a computable index for $\psi_3$ if $e_1$ and $e_2$ are computable indexes for $\psi_1$ and $\psi_2$, respectively. These insights, which are not explicitly stated, are needed to prove that $t:\mathbb{R}_c\to\mathbb{R}_c$ is given by a computable function $\phi:\mathbb{N}\to\mathbb{N}$.

Now, let us argue that there indeed is a computable function $\phi$ such that $t$ is given by $\phi$. Let us give a fairly detailed argument. By the results in Section 4.1, for each fixed $k$, the function $t_k$ is computable, and given $k$, we can construct $\psi_k:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ that gives $t_k$. Moreover, given $k$, we can construct $\psi_{\Sigma k}:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ that gives $\sum_{i=1}^{k} t_i(x)$. Since $\psi_{\Sigma k}$ can be constructed uniformly in $k$, we have a computable function $\eta$ such that $\eta(k)$ is an index for $\psi_{\Sigma k}$. Let

$$\Psi(k,y,n) \;=\; \mathcal{U}(\mu t[\,\mathcal{T}_2(\eta(k),y,n,t)\,])\,.$$

By Kleene's Normal Form Theorem, we have $\Psi(k,y,n)=\psi_{\Sigma k}(y,n)$, and thus, $\Psi(k,\widehat{\alpha},n)$ is a realizer of $t_k(\{\alpha\})$. By the 1st premise above, we have a computable

function $\rho$ such that $b_k \notin \underline{\alpha(2^{\rho(\widehat{\alpha})})}$ if $k \geq \rho(\widehat{\alpha})$. By the 2nd premise $\Psi(\rho(\widehat{\alpha}), \widehat{\alpha}, n)$ will be a realizer of $t(\{\alpha\})$. Hence, let $\phi(y, n) = \Psi(\rho(y), y, n)$. Then, $t$ is given by the computable function $\phi$.

## 4.5 Exercises

**Exercise 1 (solution)**

Two (disjunct) computably enumerable sets $A_0$ and $A_1$ are *computably inseparable* if there does not exist a computable set $B$ such that $A_0 \subseteq B$ and $A_1 \subseteq \overline{B}$. Use the function $\kappa$ (see Definition 4.7) to prove that there exists two computable inseparable (but computably enumerable) sets. (Computably inseparable sets are important, e.g., such sets can be used to prove incompletess theorems for weak first-order theories.)

**Exercise 2**

Let $a, b$ be any computable real numbers such that $a < b$. Explain why there exists a computable function that does not have an upper bound on the interval $[a, b]$, but still is defined for every $x$ in the interval.

**Exercise 3 (solution)**

In this exercise you should use the function $\kappa$ (see Definition 4.7) to prove a variant of Gödel's 1st Incompleteness Theorem. You will need definitions and notation from L&K [8].

**Theorem 4.14.** *Let $\Sigma$ be a consistent and computably enumerable set of $\mathcal{L}_{NT}$-formulas such that there for any unary computable function $f$ exists an $\mathcal{L}_{NT}$-formula $\phi(x, y)$ with the property*

$$f(a) = b \quad \Leftrightarrow \quad \Sigma \vdash \forall y[\ \phi(\overline{a}, y) \leftrightarrow y = \overline{b}\ ] \qquad (*)$$

*(for all $a, b \in \mathbb{N}$). Assume that $\Sigma \vdash \overline{0} \neq \overline{1}$. Then there exists an $\mathcal{L}_{NT}$-formula $\theta$ such that $\Sigma \nvdash \theta$ and $\Sigma \nvdash \neg\theta$.*

Prove the theorem. Hint: Use that the sets $A_0$ and $A_1$ given in the soulution of exercise 4.5#1 are computably inseparable. (Standard weak $\mathcal{L}_{NT}$-theories, like Robinsom arithmetic and the theory $N$ in L&K [8], satisfy (*).)

# Chapter 5

# No Heine-Borel

## 5.1 The Heine-Borel Theorem

The next definition should make sense both fin computable and classical analysis.

**Definition 5.1.** *An* open cover *of an interval $I$ is a set of open intervals $\mathcal{A}$ such that $I \subseteq \bigcup \mathcal{A}$.*

**Example.** The set

$$\left\{ \left(-\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{2}, \frac{11}{10}\right) \right\}$$

is an open cover of the closed interval $[0, 1]$. □

**Example.** Let

$$\mathcal{A} \;=\; \left\{ \left(\frac{1}{k}, 2\right) \mid k \in \mathbb{N} \text{ and } k \geq 2 \right\}$$

Then Ais an open cover of the open interval $(0, 1)$. Observe that no finite subset of $\mathcal{A}$ can be an open cover of $(0, 1)$: Let $\mathcal{B}$ be a finite subset of $\mathcal{A}$. For any sufficiently large $n$, we have $\frac{1}{n} \in (0, 1)$, but $\frac{1}{n} \notin \bigcup \mathcal{B}$ when $n$ is sufficiently large. Furthermore, $\mathcal{A}$ is not an open cover of the closed interval $[0, 1]$ as no interval in $\mathcal{A}$ contains 0. □

**Example.** Let

$$\mathcal{A} \;=\; \left\{ \left(\frac{1}{k}, 2\right) \mid k \in \mathbb{N} \text{ and } k \geq 2 \right\} \cup \left\{ \left(-\frac{1}{8}, \frac{1}{8}\right) \right\} .$$

Then $\mathcal{A}$ is an open cover of the closed interval $[0, 1]$. Moreover, we can pick a finite subset of $\mathcal{A}$ which is an open cover of $[0, 1]$, e.g., the subset

$$\left\{ \left(-\frac{1}{8}, \frac{1}{8}\right) \;,\; \left(\frac{1}{9}, 2\right) \right\} .$$

is an open cover of $[0, 1]$. □

The Heine-Borel Theorem is an important theorem in classical analysis. For more on the theorem and its history, see [2] (this is easy reading).

**Theorem 5.2** (Heine-Borel)**.** *Let $\mathcal{A}$ be an open cover of $[0, 1]$. Then there exists a finite subset of $\mathcal{A}$ which is an open cover of $[0, 1]$.*

*Proof.* Assume for the sake of a contradiction that no finite subset of $\mathcal{A}$ is an open cover of $[0, 1]$. We say that $z$ is *reached* is there is a finite subset $\mathcal{B}$ of $\mathcal{A}$ such that $[z, 1] \subseteq \bigcup \mathcal{B}$, and we define the set $Z$ by

$$Z \;=\; \{\, z \in [0, 1] \mid z \text{ is not reached} \,\} .$$

First we observe that $Z$ is nonempty (if $Z$ is empty, then there is a finite subset of $\mathcal{A}$ that is an open cover of $[0, 1]$, contrary to our assumption). Furthermore, we observe that $Z$ has an upper bound (for any $z \in Z$, we have $z \leq 1$). Thus, by the axiom of continuity, there is a least upper bound of $Z$. Let $z_0$ denote that least upper bound.

Since $\mathcal{A}$ is an open cover of $[0, 1]$, we have $z_0 \in (a, b)$ for some interval $(a, b)$ in $\mathcal{A}$. Pick $x, y$ such that $a < x < z_0 < y < b$. Now we have $x \in Z$. Furthermore, we have $y \notin Z$, that is, $y$ is reached. Thus, by the definition of $Z$, there is a finite subset $\mathcal{B}$ of $\mathcal{A}$ such that $[y, 1] \subseteq \bigcup \mathcal{B}$. Let $\mathcal{B}'$ be $\mathcal{B}$ extended with the interval $(a, b)$. Now, $\mathcal{B}'$ is finite, $\mathcal{B}'$ is a subset of $\mathcal{A}$, and $[x, 1] \subseteq \bigcup \mathcal{B}$. Thus, we have $x \notin Z$.

This proves that $x \in Z$ and $x \notin Z$. We have reached a contradiction. □

## 5.2 No Heine-Borel

Our next theorem tells us that Theorem 5.2 restricted to intervals of $\mathbb{R}_c$ is false. We will use the notation $[\![\Phi_e(x)]\!]_{\mathbb{Q}}$. We assume that any natural number represents

(encodes) a rational number, and $[\![\Phi_e(x)]\!]_{\mathbb{Q}}$ denotes the rational number represented by $\Phi_e(x)$ (if $\Phi_e(x)$ is defined). If $\Phi_e(x)$ is undefined, then $[\![\Phi_e(x)]\!]_{\mathbb{Q}}$ is undefined.

**Theorem 5.3** (No Heine-Borel)**.** *Let $[0,1]_c$ denote the set of all computable real numbers in the interval $[0,1]$. There exists a set $\mathcal{A}$ of open intervals (with rational endpoints) such that*

$$[0,1]_c \subseteq \bigcup \mathcal{A} \tag{1}$$

*and*

$$\text{if } \mathcal{B} \text{ is a finite subset of } \mathcal{A}, \text{ then } [0,1]_c \not\subseteq \bigcup \mathcal{B}. \tag{2}$$

*Proof.* Let

$$A \;=\; \{\, e \mid \text{exists } t \text{ such that } \mathcal{T}_1(e, 2^{e+17}, t) \,\}\,.$$

Let $I_e$ be the interval

$$I_e \;=\; \left( [\![\Phi_e(2^{e+17})]\!]_{\mathbb{Q}} - \frac{1}{2^{e+16}}\,,\, [\![\Phi_e(2^{e+17})]\!]_{\mathbb{Q}} + \frac{1}{2^{e+16}} \right)\,.$$

Let $\mathcal{A} = \{I_e \mid e \in A\}$.

Assume $\alpha \in [0,1]_c$. Let $f$ be a realizer of $\alpha$, and let $e$ be a computable index for $f$. Now, $f$ is a total function. Hence, the predicate $\mathcal{T}_1(e, 2^{e+17}, t)$ holds for some $t$. This entails that the interval $I_e$ is in the set $\mathcal{A}$, and it is easy to see that $\alpha$ lies in the interval $I_e$. Hence $\alpha \in \bigcup \mathcal{A}$. This prove (1).

In order to prove (2), we consider the sum of the length of all the intervals in $\mathcal{A}$. If $I_e = (p,q)$, then we have $q - p = \frac{2}{2^{e+16}} = \frac{1}{2^{e+15}}$. Let $L$ be the sum of the length of all intervals in $\mathcal{A}$. Then we have

$$L \;<\; \sum_{e=0}^{\infty} \frac{1}{2^{e+15}} \;=\; \frac{1}{2^{14}}\,.$$

Let $\mathcal{B}$ be a finite subset of $\mathcal{A}$. It is easy to see that $[0,1]_c \not\subseteq \bigcup \mathcal{B}$ (and if you do not think it is easy, you can go on and read the rest of this proof).

Assume

$$\mathcal{B} \;=\; \{\; (a_1,b_1), (a_2,b_2), \ldots, (a_k,b_k) \;\} \;\subseteq\; \mathcal{A}\,.$$

We will prove that there exists a number $c$ in the set $\{b_1, b_2, \ldots, b_k, 0\}$ such that $c \in [0,1]_c$ and $c \notin \bigcup \mathcal{B}$ (and hence we have $[0,1]_c \not\subseteq \bigcup \mathcal{B}$). Consider the following procedure:

**Step 1.** Pick an interval $(a_{i_1}, b_{i_1})$ in $\mathcal{B}$ such that $0 \in (a_{i_1}, b_{i_1})$. If no such interval exists, we have $0 \in [0,1]_c$ and $0 \notin \bigcup \mathcal{B}$. If such an interval exists, proceed to Step 2.

**Step 2.** We have $b_{i_1} \in [0,1]_c$ as $L < 1/2^{14}$. Pick an interval $(a_{i_2}, b_{i_2})$ in $\mathcal{B}$ such that $b_{i_1} \in (a_{i_2}, b_{i_2})$. If no such interval exists, we have $b_{i_1} \in [0,1]_c$ and $b_{i_1} \notin \bigcup \mathcal{B}$. If such an interval exists, proceed to Step 3.

**Step 3.** We have $b_{i_2} \in [0,1]_c$ as $L < 1/2^{14}$. Moreover, we have $b_{i_1} < b_{i_2}$. Pick an interval $(a_{i_3}, b_{i_3})$ in $\mathcal{B}$ such that $b_{i_2} \in (a_{i_3}, b_{i_3})$. If no such interval exists, we have $b_{i_2} \in [0,1]_c$ and $b_{i_2} \notin \bigcup \mathcal{B}$. If such an interval exists, proceed to Step 4.

. . . and thus we proceed . . .

**Step $j+1$.** We have $b_{i_j} \in [0,1]_c$ as $L < 1/2^{14}$. Moreover, we have $b_{i_j} < b_{i_{j+1}}$. Pick an interval $(a_{i_{j+1}}, b_{i_{j+1}})$ . . .

If we manage to reach Step $k$, we can conclude that $b_{i_k} \in [0,1]_c$ and $b_{i_k} \notin \bigcup \mathcal{B}$. $\square$

## 5.3 Exercises

### Exercise 1

In the proof of Theorem 5.3 we define a set $\mathcal{A}$ of open intervals with rational endpoints. Let $B$ be the set of natural numbers given by

$$x \in B \;\; \Leftrightarrow \;\; (\llbracket (x)_1 \rrbracket_{\mathbb{Q}}, \llbracket (x)_2 \rrbracket_{\mathbb{Q}}) \in \mathcal{A} \, .$$

Prove that $B$ is not computable. Prove that $B$ is semi-computable.

# Chapter 6

# Solutions to Exercises

**Exercise 1.4 #2.** Assume $\langle m, n\rangle = \langle m', n'\rangle$, that is

$$m + \frac{(m+n)(m+n+1)}{2} \;=\; m' + \frac{(m'+n')(m'+n'+1)}{2}\;.$$

Thus, we also have

$$2m + (m+n)(m+n+1) \;=\; 2m' + (m'+n')(m'+n'+1)\;. \tag{6.1}$$

The proof splits into the cases (a) $m+n = m'+n'$ and (b) $m+n \neq m'+n'$.

*Case (a).* By (6.1), we have

$$m \;=\; m'\;. \tag{6.2}$$

By (6.2) and our assumption $m+n = m'+n'$, we have $n = n'$. This complete the proof of Case (a).

*Case (b).* We prove that this case is not possible. The proof splits into the two cases $m+n < m'+n'$ and $m+n > m'+n'$. Assume $m+n < m'+n'$. Then we have

$$(m'+n')(m'+n'+1) - (m+n)(m+n+1) \;\geq\; 2(m'+n') \tag{6.3}$$

By (6.1), we have

$$2m - 2m' \;=\; (m'+n')(m'+n'+1) - (m+n)(m+n+1)\;. \tag{6.4}$$

Thus

$$2m \;\geq\; 2m - 2m' \overset{(6.4)}{=} (m'+n')(m'+n'+1) - (m+n)(m+n+1)$$
$$\overset{(6.3)}{\geq} 2(m'+n') \;>\; 2(m+n) \;\geq\; 2m\,.$$

This is a contradiction. A symmetric argument yields a contradiction if $m+n > m'+n'$. Thus, Case (b) is not possible, and we have $m = m'$ and $n = n'$ whenever $\langle m, n\rangle = \langle m', n'\rangle$.

**Exercise 1.4 #4.** Let

$$f(x) \;=\; (\mu i \leq x)[\;(\exists j \leq x)[\langle i, j\rangle = x]\;]\,.$$

It is easy to see that the relation $\langle i, j\rangle = x$ is primitive recursive. The primitive recursive relations are closed under bounded quantifiers. The primitive recursive functions are closed under bounded minimalization. Thus, $f$ is a primitive recursive function. Furthermore, $\langle\cdot,\cdot\rangle$ is a bijection, and for any $x, y$ we have, $x \leq \langle x, y\rangle$ and $y \leq \langle x, y\rangle$. It follows that $f(x) = (x)_L$. Hence, $(\cdot)_L$ is a primitive recursive function.

**Exercise 1.4 #5.** By using $\langle x, y\rangle = 2^x(2y+1) - 1$ and

$$(z)_L \;=\; \mu i[\;(\;2^i \mid z+1\;) \;\wedge\; \neg(\;2^{i+1} \mid z+1\;)\;]$$

we get

$$\begin{aligned}(\langle x, y\rangle)_L \;&=\; (\,2^x(2y+1) - 1\,)_L \\ &=\; \mu i[\;(\;2^i \mid 2^x(2y+1) - 1 + 1\;) \;\wedge\; \neg(\;2^{i+1} \mid 2^x(2y+1) - 1 + 1\;)\;] \\ &=\; \mu i[\;(\;2^i \mid 2^x(2y+1)\;) \;\wedge\; \neg(\;2^{i+1} \mid 2^x(2y+1)\;)\;] \;=\; x\,.\end{aligned}$$

The last equality holds since 2 does not divide the odd number $2y+1$. Furthermore, by using

$$(z)_R \;=\; (((z+1)/2^{(z)_L}) - 1)/2$$

we get

$$\begin{aligned}(\langle x, y\rangle)_R \;&=\; (((\langle x, y\rangle + 1)/2^{(\langle x,y\rangle)_L}) - 1)/2 \;=\; (((\langle x, y\rangle + 1)/2^x) - 1)/2 \\ &=\; (((2^x(2y+1) - 1 + 1)/2^x) - 1)/2 \;=\; (2y + 1 - 1)/2 \;=\; y\,.\end{aligned}$$

**Exercise 2.7 #1.** We have to prove that $\underline{\alpha(n)}$ and $\underline{\beta(m)}$ overlap for all $n, m$, that is, we have to prove that

$$|\alpha(n) - \beta(m)| \ \leq \ \frac{1}{n+1} + \frac{1}{m+1} \ . \qquad (*)$$

It is easy to check that (*) holds when $\alpha(n) = \frac{1}{2^n}$ and $\beta(m) = \frac{1}{3^m}$. It is easy to see that we have

$$\left| \frac{1}{2^n} - \frac{1}{3^m} \right| \ \leq \ \frac{1}{2^n} + \frac{1}{3^m} \ \leq \ \frac{1}{n+1} + \frac{1}{m+1} \ .$$

**Exercise 2.7 #2.** The number is 0.

**Exercise 2.7 #3.** We have to prove that there exist $n, m$ such that $\underline{\alpha(n)}_r < \underline{\beta(m)}_\ell$, that is, we have to prove that there exist $s, t$ such that

$$\alpha(n) + \frac{1}{n+1} \ < \ \gamma(m) - \frac{1}{m+1} \ .$$

It is easy to find such $n$ and $m$, e.g., let $n = 2$ and $m = 100$. We have

$$\alpha(2) + \frac{1}{2+1} \ = \ \frac{1}{2^2} + \frac{1}{3} \ < \ \frac{3}{4} \ < \ \frac{100}{101} - \frac{1}{101} \ = \ \gamma(t) - \frac{1}{m+1} \ .$$

**Exercise 2.7 #4.** Let $\delta(s) = \frac{1}{2}$ for all $s$.

First we prove that $\delta$ is a realizer. For any $n$ and $m$, we have

$$0 \ = \ |\delta(n) - \delta(m)| \ \leq \ \frac{1}{n+1} + \frac{1}{m+1} \ .$$

This prove that $\delta$ is a realizer.

Next we prove that $\{\alpha\} < \{\delta\}$. We have to prove that there exist $n, m$ such that

$$\underline{\alpha(n)}_r \ = \ \alpha(n) + \frac{1}{n+1} \ < \ \delta(m) - \frac{1}{m+1} \ = \ \underline{\delta(m)}_\ell \ . \qquad (*)$$

Let $n = 3$. Then we have

$$\alpha(n) + \frac{1}{n+1} \ = \ \frac{1}{2^3} + \frac{1}{4} \ = \ \frac{3}{8} \ .$$

Pick any $t > 8$. Then (*) holds.

The proof that $\{\delta\} < \{\gamma\}$ is simmilar. Provide $n, m$ such that

$$\underline{\delta(n)}_r \ = \ \delta(n) + \frac{1}{n+1} \ < \ \gamma(m) - \frac{1}{m+1} \ = \ \underline{\gamma(m)}_\ell \ .$$

**Exercise 2.7 #8.** We have to prove that $\underline{\alpha(s)}$ and $\underline{\alpha(t)}$ overlap for any $s, t$ (then $\alpha$ is a realizer).

We will prove that we for any even $s$ have

$$\underline{\alpha(s+1)}_\ell \;\leq\; \underline{\alpha(s)}_\ell \;\leq\; \underline{\alpha(s+1)}_r \,. \tag{6.5}$$

and that we for any odd $s$ have

$$\underline{\alpha(s+1)}_\ell \;\leq\; \underline{\alpha(s)}_r \;\leq\; \underline{\alpha(s+1)}_r \,. \tag{6.6}$$

Now, (6.5) says that the left end point of the interval $\underline{\alpha(s)}$ lies inside the interval $\underline{\alpha(s+1)}$ if $s$ is even, and (6.6) says that the right end point of the interval $\underline{\alpha(s)}$ lies inside the interval $\underline{\alpha(s+1)}$ if $s$ is even. It follows that straightforwardly that $\underline{\alpha(s)}$ and $\underline{\alpha(t)}$ overlap for any $s, t$.

We prove (6.5) (the proof of (6.6) is symmetric). Let $s$ be even. We have

$$\underline{\alpha(s+1)}_\ell \;\leq\; \underline{\alpha(s)}_\ell \;\leq\; \underline{\alpha(s+1)}_r$$

if and only if

$$\alpha(s) - \frac{4}{2(s+1)+1} - \frac{1}{s+2} \;\leq\; \alpha(s) - \frac{1}{s+1} \;\leq\; \alpha(s) - \frac{4}{2(s+1)+1} + \frac{1}{s+2} \,.$$

It is easy to see that the inequality

$$\alpha(s) - \frac{4}{2(s+1)+1} - \frac{1}{s+2} \;\leq\; \alpha(s) - \frac{1}{s+1}$$

holds. It remains to verify that we have

$$\alpha(s) - \frac{1}{s+1} \;\leq\; \alpha(s) - \frac{4}{2(s+1)+1} + \frac{1}{s+2} \,. \tag{6.7}$$

Now, (6.7) holds in and only if

$$\frac{4}{2(s+1)+1} \;\leq\; \frac{1}{s+1} + \frac{1}{s+2} \tag{6.8}$$

and thus we are left to prove (6.8).

In order to prove (6.8), we observe that

$$\frac{1}{s+1} + \frac{1}{s+2} = \frac{(s+1)+(s+2)}{(s+1)(s+2)}.$$

and

$$\frac{4}{2(s+1)+1} = \frac{4}{(s+1)+(s+2)} .$$

Hence, (6.8) is equivalent to

$$4 \leq \frac{[(s+1)+(s+2)]^2}{(s+1)(s+2)} \tag{6.9}$$

and (6.9) holds since

$$\begin{aligned}
[(s+1)+(s+2)]^2 &= (s+1)^2 + 2(s+1)(s+2) + (s+2)^2 \\
&= \big(s^2+2s+1\big) + \big(2s^2+6s+4\big) + \big(s^2+4s+4\big) \\
&= 4s^2 + 12s + 9 \\
&\geq 4s^2 + 12s + 8 \\
&= 4(s^2+3s+2) \\
&= 4(s+1)(s+2).
\end{aligned}$$

**Exercise 2.7 #9.** It is $\pi$.

**Exercise 2.7 #11.** Let $\alpha$ be an arbitrary realizer. The realizer $0^*$ is given by $0^*(n) = 0$ (for any $n$). The realizer $(\alpha + 0^*)$ is given by

$$(\alpha + 0^*)(n) = \alpha(f_+(n)) + 0^*(f_+(n)) .$$

We need to prove that $\underline{\alpha(n)}$ and $\underline{(\alpha+0^*)(m)}$ overlap for any $n, m \in \mathbb{N}$.

We have

$$(\alpha + 0*)(m) = \alpha(f_+(m)) + 0^*(f_+(m)) = \alpha(f_+(m)) + 0 = \alpha(f_+(m)) .$$

Now, since $\alpha$ is a realizer, $\underline{\alpha(n)}$ and $\underline{(\alpha+0^*)(m)}$ overlap (it is not necessary to look at the definition of $f_+$).

**Exercise 2.7 #13.** It is trivial that (1) implies (2). We prove that (2) implies (1) by a contrapositive argument. Assume there exist $n$ and $m$ such that $\underline{\alpha(n)}$ and $\underline{\beta(m)}$ do not overlap. We can w.o.l.g. assume that $\underline{\alpha(n)}_r < \underline{\beta(m)}_\ell$. Pick $k$ such that $-\underline{\alpha(n)}_r + 2(k+1)^{-1} < \underline{\beta(m)}_\ell$. Since $\underline{\alpha(k)}$ overlap with $\underline{\alpha(n)}$ and $\underline{\beta(k)}$ overlap with $\underline{\beta(m)}$, it cannot be case that $\underline{\alpha(k)}$ and $\underline{\beta(k)}$ overlap.

**Exercise 3.6 #4.**

**Lemma 6.1.** *Let $q$ and $q'$ be rational numbers such that $q < q'$. For any $n \in \mathbb{N}$, let $\beta(n) = q$ and $\beta'(n) = q'$. Then we have $\{\beta\} < \{\beta'\}$.*

We prove the lemma. Pick $n, m$ such that

$$\frac{1}{n+1} \;=\; \frac{1}{m+1} \;<\; \frac{q'-q}{2}\,.$$

Then we have

$$\underline{\beta(n)}_r \;=\; \beta(n) + \frac{1}{n+1} \;<\; \beta'(m) - \frac{1}{m+1} \;=\; \underline{\beta'(m)}_\ell\,.$$

Thus we have $\{\beta\} < \{\beta'\}$. This concludes the proof of the lemma.

Now, assume that $\{\alpha\} < \{\alpha'\}$. Then we have $s$ and $t$ such that

$$\alpha(s) + \frac{1}{s+1} \;<\; \alpha'(t) - \frac{1}{t+1}\,.$$

Let $q' = \alpha'(t) - \frac{1}{t+1}$. Let $q_0 = (\alpha'(s) + \frac{1}{s+1} + q')/2$, and let $q_{i+1} = (q_i + q')/2$. Now we have

$$\alpha(s) + \frac{1}{s+1} \;<\; q_0 \;<\; q_1 \;<\; q_2 \;<\; \ldots q_i \;<\; q_{i+1} \ldots \;<\; q' \;=\; \alpha'(t) - \frac{1}{t+1}\,.$$

We define the function $f$ by $f(0, x) = q_0$ and

$$f(i+1, x) \;=\; \frac{f(i,x) + q'}{2}\,.$$

It is easy to that $f$ is a computable function (indeed, $f$ is defined by primitive recursion over primitive recursive functions). Moreover, we have $f(n, s) = q_n$ for all $s$. Let $g_n(x) = f(n, x)$, and let $a_n = \{g_n\}$. Then $a_0, a_1, a_2 \ldots$ is a computable sequence of computable real numbers. By Lemma 6.1, we have

$$\{\alpha\} \;<\; a_0 \;<\; a_1 \;<\; \ldots \;<\; a_i \;<\; a_{i+1} \;<\; \ldots \;<\; \{\alpha'\}\,.$$

**Exercise 3.6 #9.** By Corollary 3.19 there exists a computable real number $a$ that is not primitive recursive. We can w.l.o.g. assume that $a$ is in the interval $(0, 1)$. By Lemma 3.11 $E_2^a$ is a computable function. Let $x \in A$ iff $E_2^a(x) = 0$. Now, $A$ is a

computable set as $E_2^a$ is a total computable function, indeed, $E_2^a$ is the characteristic function of $A$.

Assume for the sake of a contradiction that $A$ is a primitive recursive set. Then $E_2^a$ is a primitive recursive function. Let

$$\alpha(n) \;=\; \sum_{i=0}^{n} E_2^a(i) b^{-i} \; .$$

Then $\alpha$ is primitive recursive as the primitive recursive functions are closed under bounded sums. It is easy to see that $\alpha$ is a realizer of $a$. But $a$ has no primitive recursive realizers. Contradiction.

**Exercise 3.6 #10.** By Corollary 3.19 there exists a computable real number $a$ that is not primitive recursive. Let $\alpha$ be a realizer of $a$. Any realizer is a total and computable function. Let $e$ be a computable index for $\alpha$. By by Kleene's Normal Form Theorem we have a primitive recursive predicate and $\mathcal{T}_1$ and a primitive recursive function $\mathcal{U}$ such that

$$\alpha(x) \;=\; \mathcal{U}(\; (\mu t)[\mathcal{T}_1(e,x,t)] \;) \; .$$

Now, let

$$f(x) \;=\; \mu t[\mathcal{T}_1(e,x,t)] \; .$$

Since $\alpha$ is a total function, $f$ will also be a total function. Assume for the sake of a contradiction that there exists a primitive recursive $g$ such that $g(x) \geq f(x)$ holds for all $x$. Then we have

$$\alpha(x) \;=\; \mathcal{U}(\; (\mu t \leq g(x))[\mathcal{T}_1(e,x,t)] \;) \; .$$

and $\alpha$ will be a primitive recursive as the primitive recursive function are closed under bounded minimalization. This contradicts that $a$ is not primitive recursive real number.

Hence for any primitive recursive $g$ we have $f(x) > g(x)$ for infinitely many $x$.

**Exercise 3.6 #11.** Modify the proof of Theorem 3.16. Let the interval $I_0 = (p,q)$. The proof will still go trough. The the computable real number $\{\beta\}$ which is constructed in the proof, will end up in the inteval $(p,q)$.

**Exercise 4.5 #1.** Let $A_0 = \{x \in \mathcal{K} \mid \kappa(x) = 0\}$ and $A_1 = \{x \in \mathcal{K} \mid \kappa(x) = 1\}$. It is easy to see that $A_0$ and $A_1$ are computably enumerable sets.

Assume there is a computable set $B$ such that $A_0 \subseteq B$ and $A_1 \subseteq \overline{B}$. Let $\chi$ be the characteristic function of $B$, that is, let

$$\chi(x) \;=\; \begin{cases} 0 & \text{if } x \in B \\ 1 & \text{if } x \notin B. \end{cases}$$

The function $\chi$ is computable since $B$ is a computable set, moreover, $\chi$ is total, that is, $\mathrm{dom}(\chi) = \mathbb{N}$. Thus, let $e$ be a computable index for $\chi$, and we have $e \in \mathrm{dom}(\Phi_e) = \mathrm{dom}(\chi) = \mathbb{N}$. Now, by the definition of $\mathcal{K}$, we have $e \in \mathcal{K}$, and thus, we have either $\kappa(e) = 0$ or $\kappa(e) = 1$.

Assume $\kappa(e) = 0$. We have

$$\kappa(e) = 0 \;\Rightarrow\; e \in A_0 \;\Rightarrow\; e \in B \;\Rightarrow\; \chi(e) = 0 \;\Rightarrow\; \Phi_e(e) = 0 \;\Rightarrow\; \kappa(e) = 1\,.$$

This proves that it cannot be the case that $\kappa(e) = 0$.

Assume $\kappa(e) = 1$. We have

$$\kappa(e) = 1 \;\Rightarrow\; e \in A_1 \;\Rightarrow\; e \in \overline{B} \;\Rightarrow\; \chi(e) = 1 \;\Rightarrow\; \Phi_e(e) = 1 \;\Rightarrow\; \kappa(e) = 0\,.$$

This proves that it cannot be the case that $\kappa(e) = 1$.

We have a contradiction.

**Exercise 4.5 #3.** By the theorem we have a formula $\phi$ such that

$$\kappa(a) = b \quad\Leftrightarrow\quad \Sigma \vdash \forall y[\ \phi(\overline{a}, y) \leftrightarrow y = \overline{b}\ ] \tag{6.10}$$

Let $A_0 = \{x \in \mathcal{K} \mid \kappa(x) = 0\}$ and $A_1 = \{x \in \mathcal{K} \mid \kappa(x) = 1\}$.

Assume $a \in A_0$. We have $\kappa(a) = 0$. By (6.10), we have $\Sigma \vdash y = \overline{0} \rightarrow \phi(\overline{a}, y)$. We have $\Sigma \vdash \overline{0} = \overline{0}$ (recall that $\overline{0} = \overline{0}$ is a logical truth). Thus, $\Sigma \vdash \phi(\overline{a}, \overline{0})$.

Assume $a \in A_1$. We have $\kappa(a) = 1$. By (6.10), we have $\Sigma \vdash \overline{0} \neq y \rightarrow \neg\phi(\overline{a}, y)$. We have assumed that $\Sigma \vdash \overline{0} \neq \overline{1}$. Thus, $\Sigma \vdash \neg\phi(\overline{a}, \overline{0})$.

Let $B = \{a \mid \Sigma \vdash \phi(\overline{a}, \overline{0})\}$. Then, since $\Sigma$ is consistent, we have $A_0 \subseteq B$ and $A_1 \subseteq \overline{B}$. Now, assume for the sake of a contradiction we have $\Sigma \vdash \theta$ or $\Sigma \vdash \neg\theta$ for any $\mathcal{L}_{NT}$-formula $\theta$. Then, for any $a \in \mathbb{N}$, we have $\Sigma \vdash \phi(\overline{a}, \overline{0})$ or $\Sigma \vdash \neg\phi(\overline{a}, \overline{0})$.

Since $\Sigma$ is a computably enumerable set, the set $\Gamma = \{\eta \mid \Sigma \vdash \eta\}$ will also be computably enumerable. Hence, for any $a \in \mathbb{N}$, either $\phi(\overline{a}, \overline{0})$ or $\neg\phi(\overline{a}, \overline{0})$ will be in the computable eneumerable set $\Gamma$. This implies that $B$ is a computable set and that contradicts that $A_0$ and $A_1$ are computably inseparable sets (see the solution of exercise 4.5#1)

Hence we conclude that there exists $a \in \mathbb{N}$ such that $\Sigma \nvdash \phi(\overline{a}, \overline{0})$ and $\Sigma \nvdash \neg\phi(\overline{a}, \overline{0})$.

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