UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Examination in:	INF-MAT2351 — Numerical calculations
Day of examination:	Friday 8. June 2012.
Examination hours:	09.00 – 13.00

This examination set consists of 3 pages.

Appendices:	None
Permitted aids:	Calculator

Make sure that your copy of the examination set is complete before you start solving the problems.

The exam consists of 12 sub problems, each carry equal weight on the total score.

Problem 1.

Consider the initial value problem: find $y = y(t)$ such that

$$\begin{aligned} y'(t) &= \left(\frac{1}{t} - 1\right) y(t) \qquad \text{for } t > 1 \\ y(1) &= 1 \end{aligned} \tag{1}$$

a) Show that

$$y(t) = te^{-(t-1)}$$

solves (1) for $t \geq 1$.

(Continued on page 2.)

b) Derive the following numerical scheme for (1): let $\Delta t > 0$, $t_n = 1 + n\Delta t$ for $n = 0, 1, 2, \ldots$, and let

$$y_0 = 1$$
$$y_{n+1} = \left(1 + \Delta t \left(\frac{1}{t_n} - 1\right)\right) y_n$$

Is this an explicit or an implicit scheme?

c) Show that $y_n \geq 0$ for all n if $\Delta t \leq 1$.

Problem 2.

Consider this system of nonlinear ordinary differential equations: find $u = u(t)$ and $v = v(t)$ such that

$$\begin{aligned} u'(t) &= v \\ v'(t) &= -f(u)u \end{aligned} \tag{2}$$

where f is some real function (of u).

a) Show that an implicit Euler discretization of (2) can be written for each step n as: find u_n and v_n such that

$$\begin{aligned} u_n - \Delta t v_n - u_{n-1} &= 0 \\ v_n + \Delta t f(u_n) u_n - v_{n-1} &= 0 \end{aligned} \tag{3}$$

b) Apply Newton's method to the nonlinear system of equations (3) and write down the linear system of equations for each Newton iteration k on the form $A_{n,k} x_{n,k} = b_{n,k}$. Show that the determinant of $A_{n,k}$ is

$$\det A_{n,k} = 1 + (\Delta t)^2 \left(f'(u_{n,k}) u_{n,k} + f(u_{n,k})\right) \tag{4}$$

c) Let $f(u) = u^2$. Show that $\det A_{n,k} \neq 0$ for all n, k. Why is this important? (Explain in one sentence.)

(Continued on page 3.)

Problem 3.

a) Compute a least squares approximation of $y(t) = t^4$ on $t \in [-L, L]$ using $p(t) = \alpha t^2$. The parameter α will depend on L, find $\alpha(L)$.

b) We have given data points $(t_i, y_i), i = 1, \ldots, n$, and want to compute a least squares approximation using $p(t) = \sum_{j=0}^{m} a_j t^j$. Write down the function F that we need to minimize to solve this problem.

c) To find the minimum we need the partial derivatives of the function found in **b)**. What is $\frac{\partial F}{\partial a_j}$?

Problem 4.

Consider the following problem: find $u = u(x,t)$ such that

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad x \in [0, L], \tag{5}$$

$$u(0,t) = u(L,t) = 0, \quad t > 0, \tag{6}$$

$$u(x,0) = f(x) \quad x \in [0, L], \tag{7}$$

where $\kappa > 0$ is the diffusion constant.

a) Write down, in pseudo code, a finite difference scheme for this problem using the explicit Euler method in time and central differences in space.

b) What is the stability requirement for the time step? Prove that the numerical solution is bounded by the initial condition if this condition is met; more precisely, show that

$$\max_i |u_i^l| \le \max_i |f(x_i)|$$

for all l, where u_i^l corresponds to the numerical approximation at point x_i and at time t^l.

c) If we disregard the initial condition, there will be infinitely many solutions to the system (5)–(6). Use separation of variables (in other words, start with the assumption that $u(x,t) = X(x)T(t)$) to find these solutions.

END