INF2080

Non-Context-Free Langugaes

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Repetition

Definition (Context-Free Grammar)

A *context-free grammar* is a 4-tuple (V, Σ, R, S) where

1. V is a finite set of *variables*
2. Σ is a finite set disjoint from V of *terminals*
3. R is a finite set of *rules*, each consisting of a variable and of a string of variables and terminals
4. and S is the *start variable*

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A language generated by a context-free grammar is a *context-free language*

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$$B \rightarrow 0 \mid 1 \mid 1B \mid 0B \mid \varepsilon$$

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All $\{0, 1\}$ words beginning and ending with the same digit

Chomsky Normal Form

Every context-free grammar can be rewritten into Chomsky normal form:

Definition (Chomsky Normal Form)

A grammar in Chomsky normal form consists only of rules of the following form:

$$S \to \varepsilon$$
$$A \to BC$$
$$A \to d,$$

where S is the start variable, A, B, C are variables with B, C distinct from S, and d is a terminal.

PDA

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Definition (PDA)

A PDA is a tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q, Σ, Γ, F are finite states and

1. Q is a set of states,
2. Σ is the input alphabet,
3. Γ is the stack alphabet,
4. $\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to \mathcal{P}(Q \times \Gamma_\varepsilon)$ is the transition function
5. $q_0 \in Q$ is the start state, and
6. $F \subseteq Q$ is the set of accepting states.

PDA

Examples:

PDA

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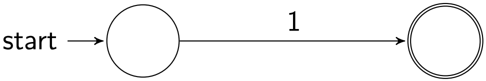

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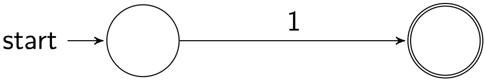
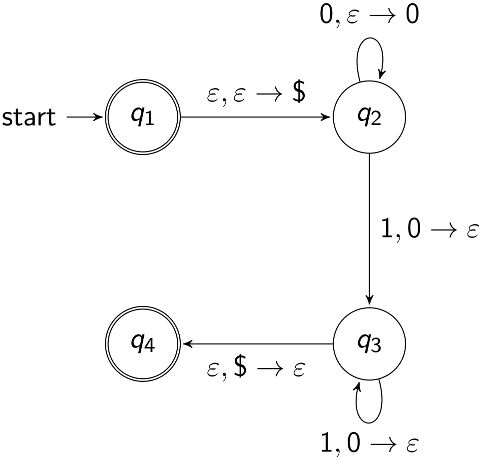

start
1


start
q_1
$\varepsilon, \varepsilon \to \$$
q_2
$0, \varepsilon \to 0$
$1, 0 \to \varepsilon$
q_3
$1, 0 \to \varepsilon$
$\varepsilon, \$ \to \varepsilon$
q_4

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Last lecture you saw that PDAs and CFLs are closely related:

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- So PDAs and CFLs have a similar relationship as NFAs and regular languages
- What about DPDAs (deterministic PDAs)? Are they the same as PDAs? $\rightarrow$ tomorrow's lecture!

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- Recall our discussion on nonregular languages, in particular the pumping lemma:

Lemma (Pumping Lemma for Regular Languages)

If A is a regular language, then there is a number p, called the pumping length, where if s is a word in A of length $\geq p$ then s can be divided into three parts, $s = xyz$, such that

1. $xy^iz \in A$ *for every* $i \geq 0$,
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- Recall that each component x, y, z corresponded to one part in an accepting path in the DFA.
- y corresponded to a cycle in the DFA
- x corresponded to a path from the initial node to the start of the cycle.
- z corresponded to a path from the end of the cycle to an accepting state.
- Condition (3) in the lemma was a useful tool when proving the nonregularity of a language.

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For every context-free language A there exists a number p (called the pumping length) where, if s is a word in A of length $\geq p$, then s can be divided into five parts, $s = uvxyz$, satisfying the following conditions:

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$\rightarrow$ Things are a bit more involved this time!

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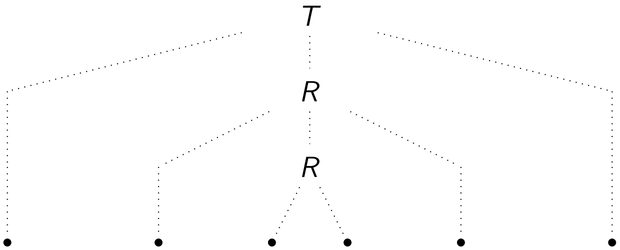
- Proof idea: Let G be the CFG generating A and τ a parse tree of s. If s is sufficiently large, we can argue that, along a path, a variable must occur twice.

Pumping Lemma - Proof

Parse tree for s:



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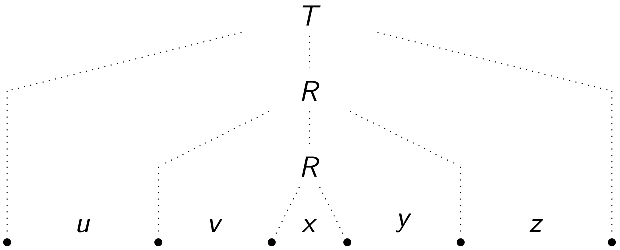

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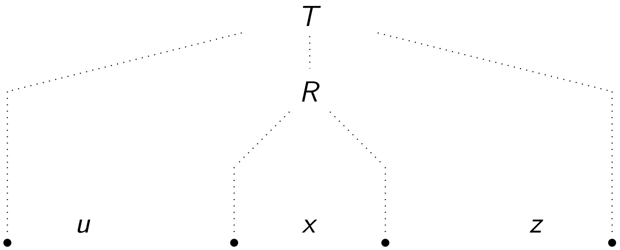

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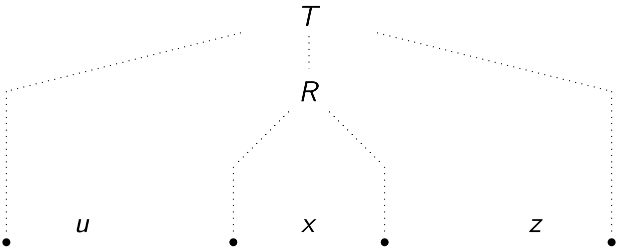

$\rightarrow uv^0xy^0z = uxz$

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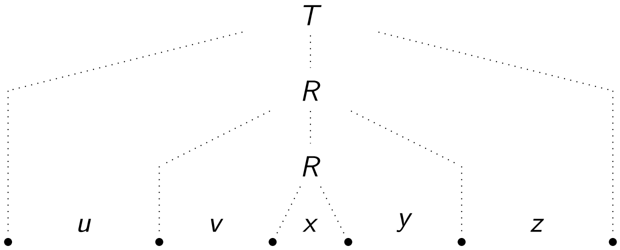

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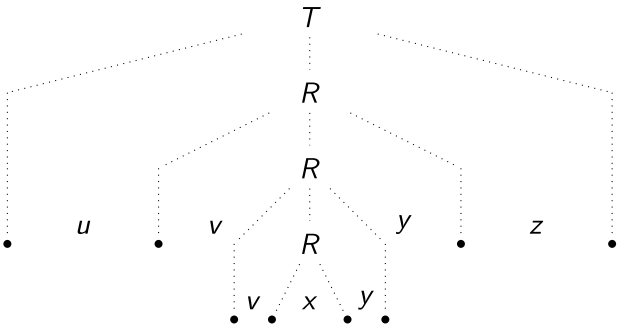

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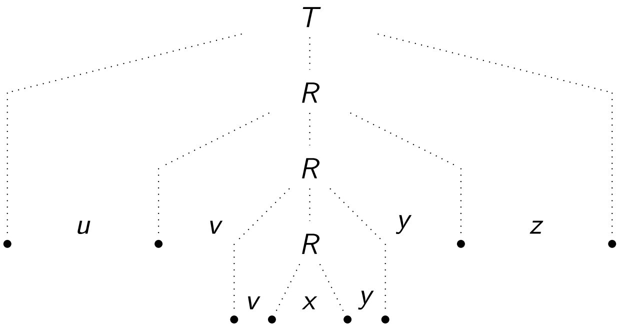

$\rightarrow uv^2xy^2z$, and so on

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- $\Rightarrow$ If a parse tree has height h, then it has at most b^h leaves.

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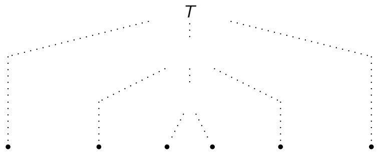

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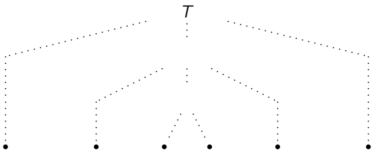

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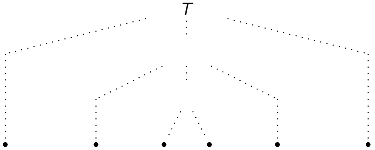

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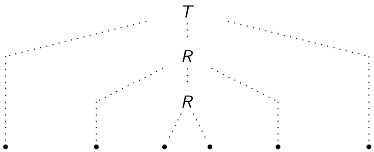

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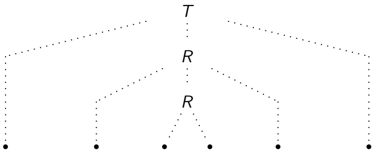

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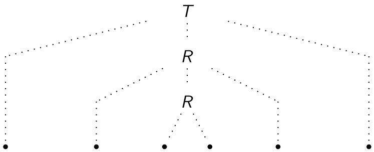

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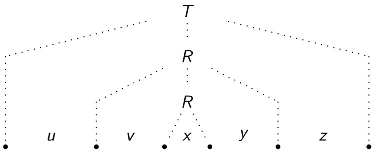

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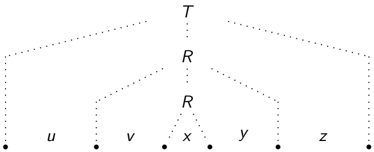
Proof:

- The first occurence of R generates vxy, while the second generates x (remember, only using rules from G),



Pumping Lemma - Proof

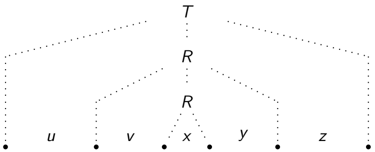
Lemma (Pumping Lemma for CFLs)

CFL A, $s \in A$ with $|s| \geq p$, then $s = uvxyz$, such that:

1. $uv^ixy^iz \in A$ *for all* $i \geq 0$,
2. $|vy| > 0$,
3. $|vxy| \leq p$.

Proof:

- The first occurence of R generates vxy, while the second generates x (remember, only using rules from G),
- Thus we can replace one subtree with the other and get valid parse trees for uxz and uv^ixy^iz, $i > 1$.



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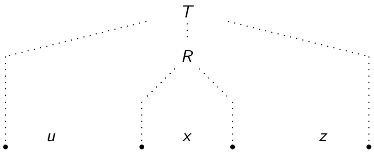

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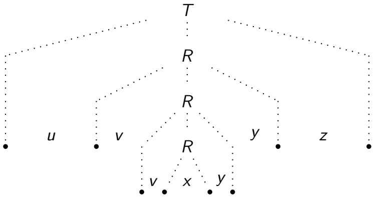

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- The first occurence of R generates vxy, while the second generates x (remember, only using rules from G),
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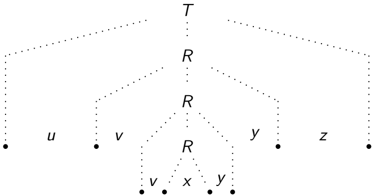
- This proves condition 1.



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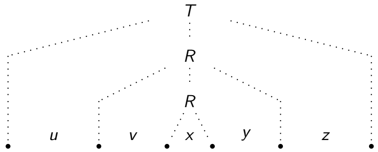

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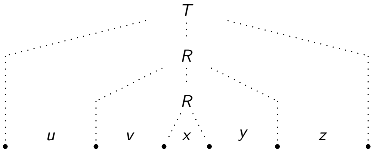
- Then replacing the first R (generating vxy) with the second R (generating x) gives us a *smaller* parse tree for s.



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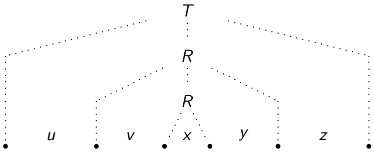

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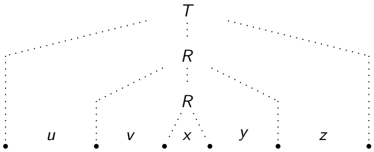

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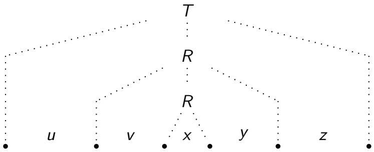

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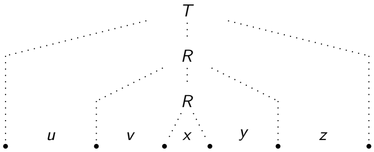
- Thus that subtree has at most $b^{|V|+1}$ leaves.



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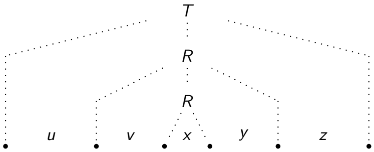

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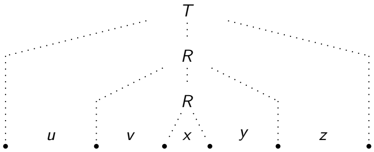

Pumping Lemma - Examples

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1. $uv^ixy^iz \in A$ *for all* $i \geq 0$,
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Language: $A_1 = \{a^nb^nc^n \mid n \geq 0\}$.

- Assume A_1 is context-free.

Pumping Lemma - Examples

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- $|s| = 3p \geq p$ and $s \in A_1$

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- Assume A_1 is context-free. Then let p be its pumping length and $s = a^pb^pc^p$.
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- Now we consider how we can subdivide s into $uvxyz$.
- Condition 2: v or y must be nonempty.

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- Then uv^2xy^2z does not have the symbols in the correct order.

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- Case 2: Both v of y contains only one type of symbol.
- Then v and y cannot contain both a's and b's or both b's and c's.
- But then uv^2xy^2z cannot have an equal number of a's, b's, *and* c's.

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- s cannot be pumped.

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1. $uv^ixy^iz \in A$ *for all* $i \geq 0$,
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Language: $A_2 = \{a^ib^jc^k \mid 0 \leq i \leq j \leq k\}$.

- Same as before, assume A_2 is context-free and let $s = a^pb^pc^p$. Again, $s = uvxyz$.

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- Case 1: Same argument as before. If either v or y contains more than one type of symbol, then uv^2xy^2z will not have the right order of symbols.

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- Case 2: If both v and y contain only one type of symbol, then we cannot use the same reasoning as in the last example (the number of a's, b's, and c's are different now!)
- However, since each contains only one type of symbol, one symbol a, b, or c does not occur in v or y.

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- Case 2a: a does not appear.

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- Case 2a: a does not appear. Then consider the string uxz. This has the same amount (p) of a's as s, but fewer b's and c's.

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- Case 2b: b does not appear.

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- Case 2b: b does not appear. Then either a's or c's must occur in v or y (both cannot be empty). If a's appear, then uv^2xy^2z has more a's than b's.

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- Case 2c: c does not appear.

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- Case 2c: c does not appear. Then the string uv^2xy^2z contains more a's or b's than c's, contradiction!

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Language: $A_3 = \{\omega\omega \,|\, \omega \in \{0,1\}^*\}$.

- Assume A_3 is CF, and choose $s = 0^p1p^p1$.

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- But this can be pumped! uv^ixy^iz is still a word in A_3.

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- But this can be pumped! uv^ixy^iz is still a word in A_3.
- We need to be more careful with our choice of s.

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- Let $s = 0^p1^p0^p1^p$. We will make use of condition 3, so $|vxy| \leq p$.

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- Let $s = 0^p1^p0^p1^p$. We will make use of condition 3, so $|vxy| \leq p$.
- Two options: vxy is contained in the first or second half of s, or covers the middle.

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- If vxy is contained in the first half, then pumping up to uv^2xy^2z "pushes" a 1 to the first position of the second half.

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Pumping Lemma - Examples

Lemma (Pumping Lemma for CFLs)

1. $uv^ixy^iz \in A$ *for all* $i \geq 0$,
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- Hence s cannot be pumped. $\Rightarrow$ A_3 is not context-free