INF3170/4170 Spring 2005:

Completeness for First-Order Tableaux

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The first-order Model Existence Theorem, as stated and proven in [1], is elegant and strong. By reading and understanding the proof, one learns a powerful proof technique and gains useful insight into core subjects of first-order logic. The proof might, however, be somewhat hard to grasp for a first time logic student. It is my fear that technical details of the proof draw the student's attention away from the primary subjects of this course; correctness and completeness of proof systems. The Model Existence Theorem is stronger than what we need to show completeness for first-order tableaux. Therefore, the completeness proof in this paper replaces the Model Existence Theorem in the syllabus for INF3170/4170 Spring 2005.

The reader should be familiar with *first-order tableaux* as defined in [1] and the concepts *tableau construction rule* (Definition 7.8.3 in [1]), *sequence of tableaux* (Definition 7.8.4 in [1]), *König's Lemma* (Lemma 2.7.2 in [1]) and *Hintikka's Lemma* (Proposition 5.6.2 in [1]).

Before we state and prove the completeness theorem we need to define what it means for a tableau construction rule to be *fair*.

1 Definition (Fairness) *A tableau construction rule $\mathcal{R}$ is* fair *provided, for any sentence Φ of any first-order language $\mathcal{L}$, the sequence $\mathbf{T}_1, \mathbf{T}_2, \ldots$, of tableaux for Φ constructed according to $\mathcal{R}$ has the following properties:*

1. *Every non-literal formula occurrence in $\mathbf{T}_n$ eventually has the appropriate Tableau Expansion Rule applied to it, on each branch on which it occurs.*
2. *For every γ-formula occurrence γ in $\mathbf{T}_n$ and every closed term t of $\mathcal{L}^{par}$, γ eventually has the γ-rule applied to it, introducing the sentence $\gamma(t)$ on each branch on which γ occurs.*

2 Theorem (Completeness) *Let X be a first-order sentence of a first-order language $\mathcal{L}$. If X is valid, then X has an atomically closed tableau proof.*

Proof We show the contrapositive, i.e. that X is *not* valid from the assumption that X has *no* atomically closed tableau proof. In order to show that X is not valid, we show that $\neg X$ is satisfiable.

Observe that $\neg X$ occurs in every branch in every tableau we can construct for $\{\neg X\}$. If we show that some branch in a tableau constructed for $\{\neg X\}$ is satisfiable, it follows that $\neg X$ is satisfiable. Let $\mathcal{R}$ be some *fair* tableau construction rule and let $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \ldots$ be a sequence of tableaux for X constructed according to $\mathcal{R}$. We assume that the sequence is infinite. Otherwise, the proof is similar, but simpler. Since the sequence is infinite, and the tree $\mathbf{T}_{n+1}$ extends[1] $\mathbf{T}_n$, we can view this process as constructing a sequence of approximations of an infinite tree. Let us call this limit object $\mathbf{T}$.

We will now construct a satisfying model for $\neg X$ based on a branch in $\mathbf{T}$ which contains no atomic contradiction, i.e. neither $\perp$, nor both A and $\neg A$, where A is an atomic sentence. Alas, we cannot directly use the assumption that X has no atomically closed tableau proof to conclude that $\mathbf{T}$ has a branch with this property. Tableaux are defined as finite objects in [1], and $\mathbf{T}$ is an infinite tree. In order to show that $\mathbf{T}$ contains a branch without atomic contradictions, we assume that $\mathbf{T}$ is atomically closed[2] and derive a contradiction. Prune each branch of $\mathbf{T}$ by removing all sentences below the occurrence of an atomic contradiction. Call the resulting tree $\mathbf{T}^*$. In $\mathbf{T}^*$, every branch is finite, so by König's Lemma (Lemma 2.7.2 in [1]) $\mathbf{T}^*$ itself is finite. Then, for some n, $\mathbf{T}^*$ must be a subtree[3] of $\mathbf{T}_n$, and thus $\mathbf{T}_n$ is atomically closed. But this is impossible, since we assume that X has no atomically closed tableau proof. Thus, $\mathbf{T}$ cannot be atomically closed.

Let θ be some branch in $\mathbf{T}$ without atomic contradictions. Since $\mathcal{R}$ is *fair*, if α occurs on θ, then so does α_1 and α_2. If β occurs on θ, then so does either β_1 or β_2. If $\neg\neg Z$ occurs in θ, then so does Z. If γ occurs on θ, then so does $\gamma(t)$ for every closed term t (of $\mathcal{L}^{par}$). If δ occurs on θ, then so does $\delta(p)$ for some parameter p of par. Since θ is without atomic contradictions, not both an atomic formula and its negation occur on θ. It follows that the set S of sentences on θ is a first-order Hintikka set (with respect to $\mathcal{L}^{par}$). By Hintikka's Lemma (Proposition 5.6.2 in [1]) S is satisfiable. Since $\neg X$ is in S, $\neg X$ is satisfiable. ∎

References

[1] M. C. Fitting. *First-Order Logic and Automated Theorem Proving.* Graduate Texts in Computer Science. Springer-Verlag, Berlin, 2nd edition, 1996. 1st ed., 1990.

[1] ($\mathbf{T}_{n+1}$ is the result of one application of a Tableau Expansion Rule to $\mathbf{T}_n$.)

[2] We define an infinite tree as atomically closed in the same way as an atomically closed tableau, i.e. that every branch contains an atomic contradiction.

[3] A tree T' is a *subtree* of a tree T if the nodes and edges of T' form subsets of the nodes and edges of T.