Assignment 2 - INF3170/4170 spring 2005
Due: 3pm, May 23rd, 2005

1 Syntax

Recall the definition of first-order language in Chapter 5.1 of Fitting's textbook. The terms and formulas of a language are defined inductively from the symbols of the language. The definition of formulas is based on the definition of terms.

When we show general properties of objects like formulas and terms, we often do so by induction on the structure of the objects. For example, if we are to show that all terms have balanced parentheses, we show it by structural induction on terms. In the base case, we look at the simplest terms; constants and variables. They have no parentheses, so all parentheses in them are balanced. In the induction step, we look at an arbitrary term $f(t_1, \ldots, t_n)$. The trick is to find a way to apply the induction hypothesis. In this case, it is quite simple, since the term contains $t_1, \ldots, t_n$ as *subterms*, on which we can use the induction hypothesis. We assume that each term t_i has balanced parentheses. At this point, we often have to do some thinking in order to conclude that the whole object has the desired property. In the above example, this is also simple, since we have added both a left and a right parenthesis.

Exercise 1

Let $FV(A)$ be the set of variables occurring free in the formula A. Let σ be a substitution with finite support D (see Fitting def. 5.2.6), and suppose $\sigma(x)$ is a ground term for every $x \in D$. Show that if A is a formula, then so is $A\sigma$, and that $FV(A\sigma) = FV(A) \setminus D$. (Remember: If B is a set, then $B \setminus D$ denotes a set difference.) This may seem obvious, but you are going to show the proposition by induction. You have to show that all terms have a similar property, and then show the proposition for all formulas! ■

Exercise 2

Give the simplest language possible of which the following five expressions are formulas. Can the formulas be interpreted naturally as true statements about numbers?

A1: $\forall x \forall y\, (Sx \doteq Sy \to x \doteq y)$

A2: $\forall x\, (\neg(x \doteq 0) \to \exists y\, (x \doteq Sy))$

A3: $\forall x\, \neg(0 \doteq Sx)$

A4: $\forall x\, (0 \dot{+} x \doteq x)$

A5: $\forall x \forall y\, (x \dot{+} Sy \doteq S(x \dot{+} y))$

■

Remark: We indicate that $\dot{+}$ and $\doteq$ are symbols of the *object language* by superscripting them with a dot. In this way, we can separate them from $+$ and $=$, which are symbols of the *meta language*. We could for instance have written @ for $\dot{+}$ and ? for $\doteq$ in the object language, but then the intended interpretation of the symbols would have been hidden.

2 Semantics

We give a model $\langle D, I \rangle$ (like in Def. 5.3.1. of the text book) satisfying the formulas in the previous exercise. This is called the *standard model*.

- The domain D is $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- S^I is the successor function, i.e. $S^I : \mathbb{N} \to \mathbb{N}$ given by $S^I(n) = n + 1$.
- $\doteq^I$ is the identity relation, i.e. $\{(0,0), (1,1), (2,2), \ldots\}$.
- $\dot{+}^I$ is the addition function, i.e. $\dot{+}^I : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $\dot{+}^I(m, n) = m + n$.

Exercise 3

You are now going to work further with the language from the previous exercise – let us call it $\mathcal{L}$. If we want to express the smaller-than relation in $\mathcal{L}$, we can do this without introducing a symbol for it in $\mathcal{L}$. Instead we can use the symbols for addition and identity and encode the smaller-than relation, i.e. we can find a formula $\Phi(t_1, t_2)$ such that $\Phi(t_1, t_2)$ is true in the standard model if and only if $t_1^I < t_2^I$. Find an expression for the formula $\Phi(t_1, t_2)$. ■

Note: Even if we can encode smaller-than, there are still many natural statements about numbers which we cannot express in this language, no matter how good we are at encoding. For example we cannot encode multiplication.

Exercise 4

Write a formula of the language $\mathcal{L}$ which contains at least three connectives, is true in the standard model, and is invalid. Show that it is not valid by giving a countermodel. ■

3 Consistency

Let $\vdash \varphi$ mean that there is a tableau proof of φ and $\nvdash \varphi$ mean that there is no tableau proof of φ. A tableau calculus is *consistent* if $\nvdash \bot$. (Recall: A set of formulas is consistent if there is no closed tableau for it.) A tableau calculus is *complete* if it proves every valid formula.

Exercise 5

Show that $\vdash \bot \to \varphi$ for all formulas φ. ■

Exercise 6

Show that a tableau calculus is consistent if and only if there is a formula φ such that $\nvdash \varphi$. HINT. You may freely use the *tableau cut rule*: At any point in a tableau construction, you can split a branch and introduce the sentences ψ and $\neg\psi$. ■

Exercise 7

Show that a tableau calculus is complete if and only if every consistent set of formulas is satisfiable. ■