Homework #2

For Friday, January 28

1. A set of sets of the form $\{A_i \mid i \in I\}$ is often referred to as a family of sets indexed by I. A standard notation the union of such a family is

$$\bigcup_{i \in I} A_i = \bigcup \{A_i \mid i \in I\} = \{a \mid a \in A_i \text{ for some } i \in I\}$$

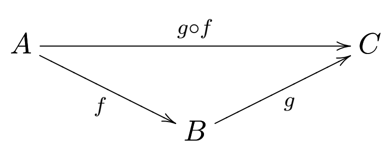
Analogous notation is used for intersection:

$$\bigcap_{i \in I} A_i = \bigcap \{A_i \mid i \in I\} = \{a \mid a \in A_i \text{ for all } i \in I\}$$

   a. Express $\mathbb{N}$ as a union of a family of singletons (sets with one element).
   b. Express $\mathbb{N}$ as a union of a *chain* of finite sets, i.e. a family $\{A_n \mid n \in \mathbb{N}\}$ such that (A_n is finite for all n and) $m \leq n$ implies $A_m \subseteq A_n$.
   c. Express $\{0\}$ as an intersection of infinite subsets of $\mathbb{N}$.

2. a. We say that a subset of natural number, $A \subseteq \mathbb{N}$ is *closed under addition* if $m, n \in A$ implies that $m + n \in A$. Give two examples of such subsets.
   b. For some property P, we say that a set A is the *least set* satisfying P if P is true of A and for any set B, if P is true of B then $A \subseteq B$. Show that there exists a least subset of $\mathbb{N}$ which contains 2 and which is closed under addition. (Hint: consider the intersection of all subsets of $\mathbb{N}$ which contain 2 and are closed under addition.)

3. Consider the composition of two functions:



   a. Show that if f and g are injections then $g \circ f$ is an injection.
   b. Show that if f and g are surjections then $g \circ f$ is a surjection.
   c. Show that if $g \circ f$ is an injection then f is an injection.

d. Show that if $g \circ f$ is a surjection then g is a surjection.

4. Show that composition of functions is *associative*, that is, for functions $f : A \to B$, $g : B \to C$, and $h : C \to D$ show that

$$h \circ (g \circ f) = (h \circ g) \circ f$$

5. We said that a function is bijective if it is both a surjection and an injection. A function $f : A \to B$ is *invertible* if there exists a function $g : B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$ (where 1_A is the identity function on A. g is called the *inverse* of f and is often denoted f^{-1}.) Show that a function is bijective if and only if it is invertible.

6. Read chapter 2 of Avigad.