Homework #3

For Friday, February 4

This is probably more than you have time for. Do at least the starred ($\star$) ones. A circle ($\circ$) next to a problem means that this problem is for your edification and entertainment. You should at least have a good look at the remaining problems, and do them if you have time; they are fair game for the final exam.

1. Read section 1.1 of the van Dalen text.

2. Use the least element principle to prove the induction principle, and vice-versa.

$\star$ 3. Prove by induction that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$. Keep in mind that $\sum_{i=0}^{n} 2^i$ is an abbrevation for $2^0 + 2^1 + 2^2 + \ldots + 2^n$.

$\circ$ 4. Can you find a formula for $\sum_{i=0}^{n} i^2$?

5. Prove by induction that whenever $n \geq 4$, $n! > 2^n$. Recall that $n!$, read "n factorial," is defined to be $n \cdot (n-1) \cdot \ldots \cdot 1$.

$\star$ 6. Prove by induction that whenever $n \geq 5$, $2^n > n^2$. (Hint: you will have to prove an auxilliary statement first.)

$\star$ 7. A "binary string of length n" is a sequence of n 0's and 1's; for example, 011101 is a binary string of length 6. Prove by induction that for every n there are 2^n binary strings of length n. How many binary strings are there having length *at most* n? Justify your answer.

8. Prove that there are 2^n subsets of a set having n elements. (Hint: you can use the preceding problem.)

$\star$ 9. Let "HiLo" be the following children's game: Player 1 picks a natural number between 1 and M (inclusive), and Player 2 tries to guess it. After each incorrect guess, Player 1 responds "higher" or "lower." Assuming Player 2 has n guesses, what is the largest value of M for which there is an algorithm that guarantees success? Describe the algorithm, and use induction to prove that it works.

◦ 10. Show that the algorithm you gave in response to the previous question is optimal, i.e. for larger values of M there will be numbers for which the algorithm fails to determine the correct number after n guesses.

11. Use the least element principle to prove that all numbers are interesting.