UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Eksamen i	INF3470/4470 — Digital signalbehandling
Eksamensdag:	11. desember 2015
Tid for eksamen:	09.00 – 13.00
Oppgavesettet er på 16 sider.	
Vedlegg:	Ingen
Tillatte hjelpemidler:	Ingen

Kontroller at oppgavesettet er komplett før du begynner å besvare spørsmålene.

Merknad 1: Alle størrelser og figurakser skal benevnes.
Merknad 2: Les gjennom hele oppgavesettet før du begynner!

Svar:

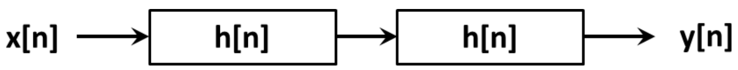
Forslag til fasit, versjon-01:

Oppgave 1 FIR filtre

Vi studerer et realiserbart FIR filter av orden M med impulsrespons $h[n]$. Passbånd- og stoppbåndfrekvensene er $\omega_p = 0.2$ rad/sample og $\omega_s = 0.3$ rad/sample. Pass- og stopbåndets rippelnivåer er $\delta_p = 0.125$ og $\delta_s = 0.1$

a)
- Hva er bredden til filterets transisjonsbånd? 0.5 p.
- Skisser magnituden til filterets frekvensrespons og indiker δ_p og δ_s på den vertikale aksen. 0.5 p.

b) Utgangen $y[n]$ fås fra inngangen $x[n]$ ved å sette to filtre med impulsrespons $h[n]$ i kaskade som vist i figuren under.



Avgjør ordenen til det effektive filteret med impulsrespons $h_{tot}[n]$ som gir $y[n]$ fra $x[n]$. Begrunn svaret. 1 p.

c) De absolutte spesifikasjonene for $h[n]$ som tilsvarer δ_p og δ_s er $A_p = 2.2$ dB og $A_s = 21$ dB. Hva er de absolutte spesifikasjonene for passbånd og stoppbånd rippelnivå for det effektive filteret beskrevet i b)? 1 p.

d) Frekvensresponsene fra $h[n]$ og $h_{tot}[n]$ er $H(e^{j\omega})$ og $H_{tot}(e^{j\omega})$.

- Gi et utrykk for fasen til $H_{tot}(e^{j\omega})$ som en funksjon av fasen til $H(e^{j\omega})$. 0.5 p.

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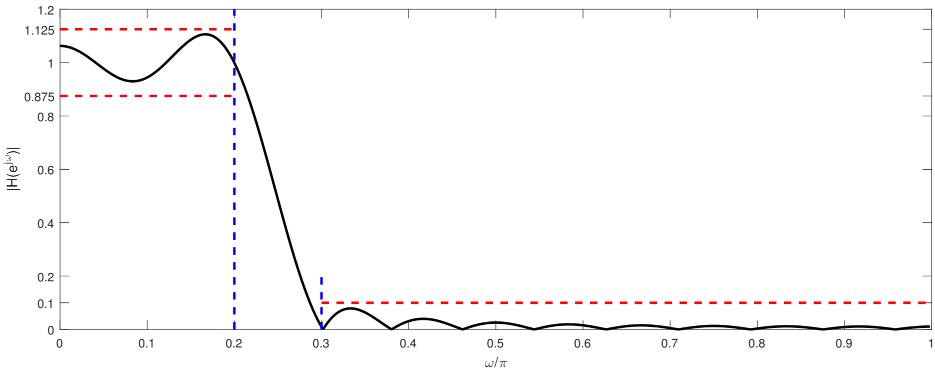
- Hvis $H(e^{j\omega})$ er lineær fase, hva kan man si om fasen til $H_{tot}(e^{j\omega})$? Begrunn svaret. 0.5 p.

Svar:

a) The width of the transition band is $\Delta\omega = \omega_s - \omega_p = 0.3 - 0.2 = 0.1$ rad/sample.



b)
- First method: $h[n]$ has length $L = M + 1$ the convolution of two signals of length L_1 and L_2 is of length $L_1 + L_2 - 1$. Therefore h_{tot} has length $2L - 1 = 2M + 1$ and the effective filter is of order $2M$
- Second method: The Z-transform $H(z)$ is a polynomial of order M ($M + 1$ coefficients). The Z-transform of the effective filter is $H_{tot}(z) = H(z)^2$ is a polynomial of order $2M$ ($2M+1$ coefficients) hence the order of the effective filter is $2M$.

c) The frequency response of the effective filter is $H_{tot}(e^{j\omega}) = H(e^{j\omega})^2$ since a convolution in time is equivalent to a product in frequency. In decibel, we then have $H_{tot}(e^{j\omega})$ [dB] $= 2H(e^{j\omega})$ [dB]. The bandpass and stopband ripple levels of the effective filter are therefore

$$A_{p_{tot}} = 2A_p = 4.4 \text{ dB}$$
$$A_{s_{tot}} = 2A_s = 42 \text{ dB}$$

d)
- Let us denote $\Psi(\omega)$ and $\Psi_{tot}(\omega)$ the phase of $H(e^{j\omega})$ and $H_{tot}(e^{j\omega})$, respectively. We have

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\Psi(\omega)}$$

and

$$H_{tot}(e^{j\omega}) = |H_{tot}(e^{j\omega})|e^{j\Psi_{tot}(\omega)} = |H(e^{j\omega})|^2 e^{j2\Psi(\omega)}$$

which gives $\Psi_{tot}(\omega) = 2\Psi(\omega)$.
- If $H(e^{j\omega})$ is linear phase, this means that $\Psi(\omega)$ is proportional to ω and $2\Psi(\omega) = \Psi_{tot}(\omega)$ is also proportional to ω. $H_{tot}(e^{j\omega})$ is consequently also linear phase.

(Fortsettes på side 3.)

Oppgave 2 Sampling og aliasing

La $x(t)$ være et signal med frekvensspektrum

$$X(\Omega) = 1 \quad \text{for } |\Omega| \leq 200\pi \text{ rad/s}$$
$$X(\Omega) = 0 \quad \text{for } |\Omega| > 200\pi \text{ rad/s}$$

a) Skisser frekvensspekteret til $x[n]$, den ideelt samplede versjonen av $x(t)$ med samplingfrekvens $F_s = 300$ Hz. 1 p.

b)
- Hvis $x[n]$ er en impulsrespons, hva slags filter tilsvarer det? 0.5 p.
- Er $x[n]$ i så fall realiserbart? Begrunn svaret. 0.5 p.

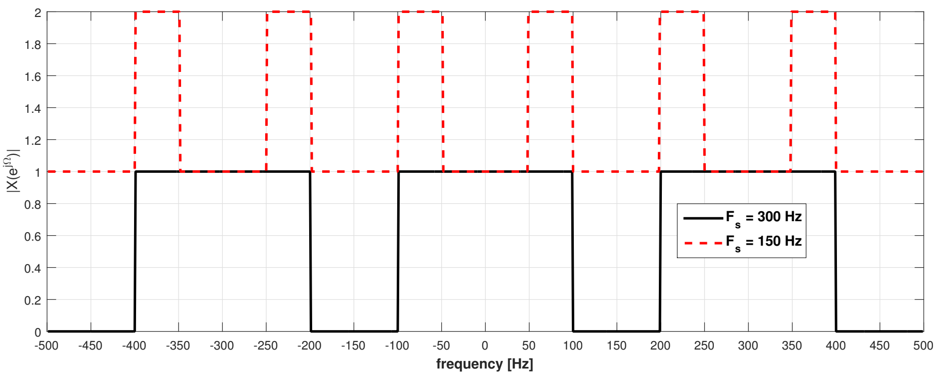
c) Skisser frekvensspekteret til $x[n]$, den ideelt samplede versjonen av $x(t)$ med samplingfrekvens $F_s = 150$ Hz. 1 p.

Svar:

a) The maximum frequency contained in $x[n]$ is 100 Hz. The sampling frequency $F_s = 300$ Hz satisfies the Shannon criterion ($F_s = 300 > 2 * 100$ Hz). There is no aliasing. See figure below.

b)
- if $x[n]$ was a filter it would be an ideal low pass filter.
- It is not realizable due to its infinitely small width for the transition band and because it nulls out all frequencies between 100 Hz and 200 Hz ($F_s - 100$). Its impulse response would also be a sinc function infinitely long and not absolutely summable.

c) The sampling frequency $F_s = 150$ Hz does not satisfy the Shannon criterion ($F_s = 150 < 2 * 100$ Hz). There is aliasing. See figure below.



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Oppgave 3 Transformanalyse av LTI systemer

Du er nylig ansatt på et forskningsprosjekt for pulsmåling av eksamenskandidater ved UiO. Dere har utviklet et kombinert sensor og dataloggingssystem som drives fra lysnettet, men dere har slurvet med hardwaredesignet og sliter veldig med mye 50 Hz målestøy fra lysnettet. Sjefen din har oppdaget at du har tatt kurs i digital signalbehandling, og vil at du skal lage et filter for å få bort denne støyen. Du synes vagt å huske at det var noe som heter notch-filter, men du var så urutinert at du solgte den gamle læreboka di. Etter desperat googling finner du følgende transferfunksjon for et notch-filter:

$$H(z) = b_0[1 - (2\cos(\phi))z^{-1} + z^{-2}] \tag{1}$$

a) Finn nullpunktene til $H(z)$ som funksjon av ϕ. Gitt at samplingsfrekvensen i systemet er $F_s = 1$ kHz, finn ϕ for å undertrykke støyen fra lysnettet. Tegn pol-nullpunktsplott for notch-filteret. 1 p.

b) Finn et utrykk for magnituderesponsen $|H(e^{j\omega})|$ til filteret og skisser denne. Du kan gjerne bruke at $b_0 = 1$ og $\cos(\phi) = 0.95$. Er filteret minimum fase? Begrunn svaret. 1 p.

c) Finn impulsresponsen og differensligningen til systemet. Anta et påtrykt inngangssignal $x[n]$. 1 p.

d) Beregn systemresponsen $y[n]$ for inngangssignalet $x[n] = s[n] + v[n]$ hvor $v[n] = A\cos(n\pi/10), n >= 0$. Kommenter det du finner. 1 p.

e) Du sliter litt med at filteret tar bort for mye av signalet i frekvensområdet rundt 50 Hz. Hva kan du gjøre for å bedre dette. Bruk pol-nullpunktsplottet og forklar kvalitativt hvordan din(e) endring(er) vil påvirke frekvensresponsen til systemet. 1 p.

Svar:

a) The zeros are found by (second order equation):

$$\begin{aligned} z_{1,2} &= \frac{2\cos(\phi) \pm \sqrt{4\cos(\phi)^2 - 4}}{2} \\ z_{1,2} &= \cos(\phi) \pm \sqrt{-\sin(\phi)^2} \\ z_{1,2} &= \cos(\phi) \pm j\sin(\phi) \\ z_{1,2} &= e^{\pm j\phi} \end{aligned} \tag{2}$$

Normalized frequency is $F = 50[Hz]/1000[Hz]$, which gives $\phi = 2\pi * 1/20 = \pi/10$. We sketch the pole/zero plot and find:

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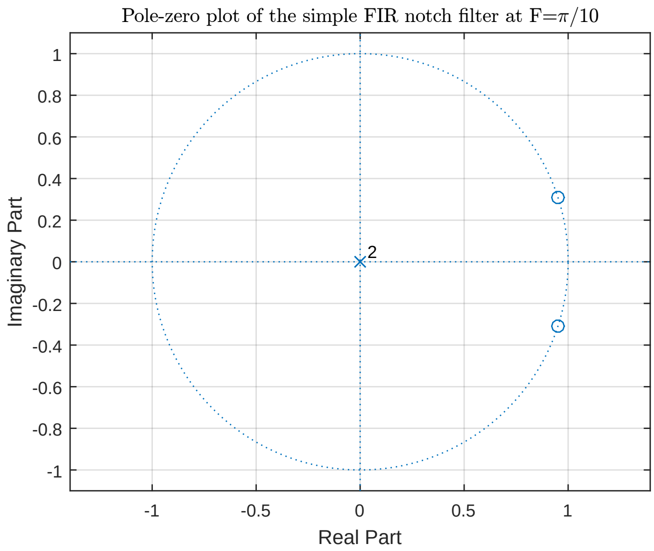


b) This exercise can be solved in several different ways. We calculate it "brute force" as follows:

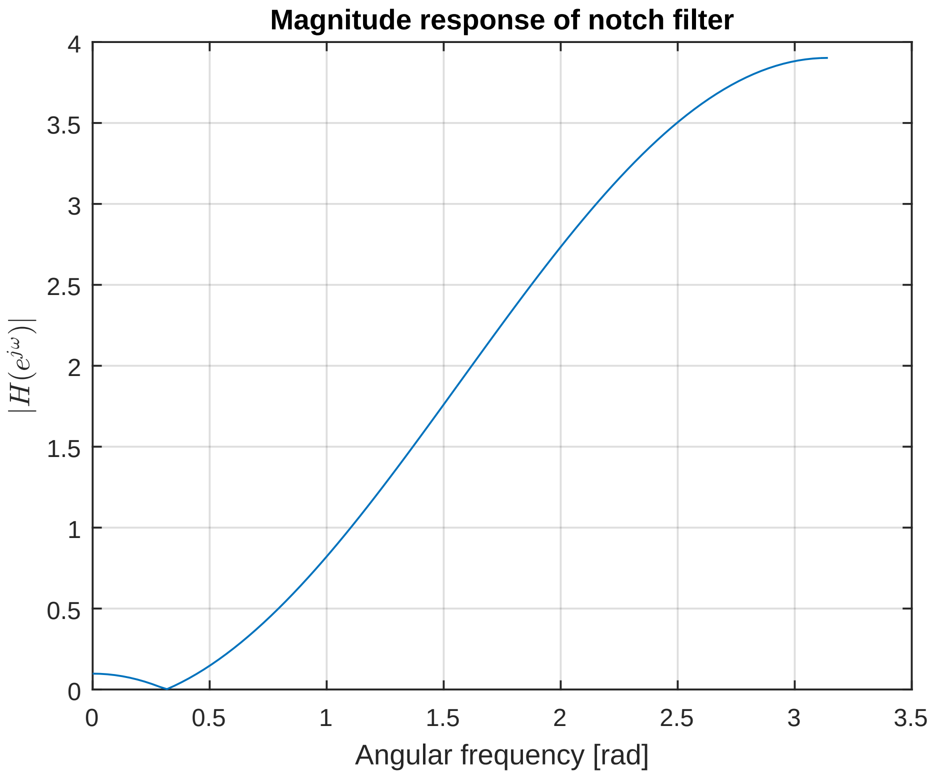
We immediately recognize this as a FIR filter, so stability is not a concern. We insert $z = e^{j\omega}$ to find the frequency response:

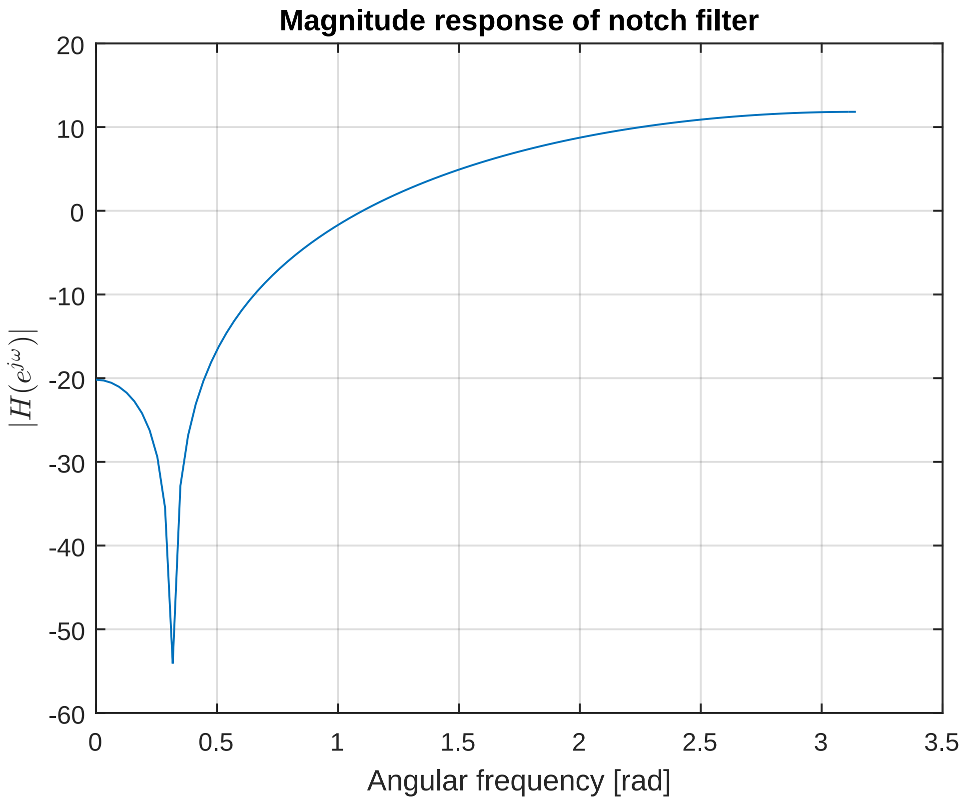
$$\begin{aligned} H(z) &= b_0[1-(2\cos(\phi))z^{-1}+z^{-2}] \\ H(e^{j\omega}) &= b_0[1-(2\cos(\phi))e^{-j\omega}+e^{-2j\omega}] \end{aligned} \tag{3}$$

There are several paths to finding the magnitude response. Here we use that $|H(e^{j\omega})|^2 = H(e^{j\omega})H(e^{-j\omega})$:

$$\begin{aligned} |H(e^{j\omega})|^2 &= b_0^2[1-2\cos(\phi)e^{-j\omega}+e^{-2j\omega}][1-2\cos(\phi)e^{j\omega}+e^{2j\omega}] \\ |H(e^{j\omega})|^2 &= b_0^2[1-2\cos(\phi)e^{j\omega}+e^{2j\omega}-2\cos(\phi)e^{-j\omega} \\ &\quad +4\cos(\phi)^2-2\cos(\phi)e^{j\omega}+e^{-2j\omega}-2\cos(\phi)e^{-j\omega}+1] \\ |H(e^{j\omega})|^2 &= b_0^2[2+2\cos(2\omega)-8\cos(\phi)\cos(\omega)+4\cos(\phi)^2] \\ |H(e^{j\omega})|^2 &= b_0^2[4\cos(\omega)^2-8\cos(\phi)\cos(\omega)+4\cos(\phi)^2] \\ |H(e^{j\omega})|^2 &= 4b_0^2[\cos(\omega)^2-2\cos(\phi)\cos(\omega)+\cos(\phi)^2] \\ |H(e^{j\omega})|^2 &= 4b_0^2(\cos(\omega)-\cos(\phi))^2 \\ |H(e^{j\omega})| &= |2b_0(\cos(\omega)-\cos(\phi))| \end{aligned} \tag{4}$$

(Fortsettes på side 6.)





There was no question about the phase response, but it could have been found as the $\tan^{-1}()$ of the real and imaginary parts of the frequency response. By using the trigonometric identities in the formula sheet, we

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would get:

$$\angle H(e^{j\omega}) = atan\left((\frac{2\cos(\phi)\sin(\omega) - \sin(2\omega)}{1 - 2\cos(\phi)\cos(\omega) + \cos(2\omega)}\right)$$
$$\angle H(e^{j\omega}) = atan\left(\frac{2\cos(\phi)\sin(\omega) - 2\sin(\omega)\cos(\omega)}{2\cos(\omega)^2 - 2\cos(\phi)\cos(\omega)}\right) \tag{5}$$
$$\angle H(e^{j\omega}) = atan\left(\frac{\sin(\omega)(\cos(\phi) - \cos(\omega))}{\cos(\omega)(\cos(\omega) - \cos(\phi))}\right)$$

Note that we could not have shortened this further immediately, as we have a 0/0 for $\omega = \phi$, which causes a discontinuity in $\omega = \phi$.

Theoretically this filter can be considered as minimum phase. The inverse is critically stable with poles on the unit circle, so it is invertible. In practice, numerical errors would probably cause an unstable inverse. You will get a score here, as long as you explain your answer correctly.

c) We use that z^{-1} can be interpreted as a delay operator and find:

$$h[n] = \begin{cases} b_0, & n = 0 \\ -2b_0\cos(\phi), & n = 1 \\ b_0, & n = 2 \\ 0, & else \end{cases} \tag{6}$$

The difference equation gets:

$$y[n] = b_0x[n] - 2b_0\cos(\phi)x[n-1] + b_0x[n-2] \tag{7}$$

d) We insert for $x[n]$ and find:

$$\begin{aligned} y[n] =& b_0(s[n] + v[n]) - 2b_0\cos(\phi)(s[n-1] + v[n-1]) + b_0(s[n-2] + v[n-2]) \\ y[n] =& (b_0s[n] - 2b_0\cos(\phi)s[n-1] + b_0s[n-2]) + \\ & (b_0v[n] - 2b_0\cos(\phi)v[n-1] + b_0v[n-2]) \end{aligned} \tag{8}$$

We cannot do much with the first term, as $s[n]$ is unknown. But the second term is more interesting. We denote it $y_v[n]$, and to simplify the notation, we write $\phi_v = \pi/10$:

$$\begin{aligned} y_v[n] &= b_0v[n] - 2b_0\cos(\phi)v[n-1] + b_0v[n-2] \\ y_v[n] &= Ab_0[\cos(\phi_vn) - 2\cos(\phi)\cos(\phi_v(n-1)) + \cos(\phi_v(n-2))] \\ y_v[n] &= Ab_0[\cos(\phi_vn) - 2\cos(\phi)\cos(\phi_vn - \phi_v) + \cos(\phi_vn - 2\phi_v)] \\ y_v[n] &= Ab_0[\cos(\phi_vn) - 2\cos(\phi)(\cos(\phi_vn)\cos(\phi_v) + \sin(\phi_vn)\sin(\phi_v)) \\ &\quad + \cos(\phi_vn)\cos(2\phi_v) + \sin(\phi_vn)\sin(2\phi_v)] \end{aligned} \tag{9}$$

(Fortsettes på side 8.)

If you answered correctly in a), you should recognize $\phi = \phi_v$, opening up for further simplification:

$$\begin{aligned}
y_v[n] &= Ab_0[\cos(\phi_v n) - 2\cos(\phi_v)(\cos(\phi_v n)\cos(\phi_v) + \sin(\phi_v n)\sin(\phi_v)) \\
&\quad + \cos(\phi_v n)\cos(2\phi_v) + \sin(\phi_v n)\sin(2\phi_v)] \\
y_v[n] &= Ab_0[\cos(\phi_v n) - 2\cos(\phi_v)^2\cos(\phi_v n) - \sin(\phi_v n)\sin(2\phi_v) \\
&\quad + \cos(\phi_v n)\cos(2\phi_v) + \sin(\phi_v n)\sin(2\phi_v)] \\
y_v[n] &= Ab_0\cos(\phi_v n)[1 - 2\cos(\phi_v)^2 + \cos(2\phi_v)] \\
y_v[n] &= 0
\end{aligned} \tag{10}$$

We thus get $y[n] = y_s[n] = b_0 s[n] - 2b_0\cos(\phi)s[n-1] + b_0 s[n-2]$, which illustrates, as expected, that the signal $v[n]$ (the 50 Hz noise) has no effect on the output signal!

e) To make the stop band narrower, a pair of poles can be positioned close to the notch filter zeros, just inside the unit circle. The closer to the unit circle, the narrower the stop band would be and more of the original input signal will be left undistorted. The notch filter is now an IIR filter.

Oppgave 4 Match systemene!

Ligninger 11 til 14 beskriver 4 systemer. Figurer 1, 2 og 3 viser tilhørende plott. I pol/nullpunktsplottene er det lagt til nullpunkter i origo for å få lik grad av teller og nevner i transferfunksjonen. Match systemene og figurene. Du kan anta at alle systemene er kausale. Angi i tillegg, når mulig, for hvert system: 5 p.

- stabilitetsegenskaper (stabilt/ustabilt/kritisk stabilt).
- faserespons (minimum fase/blandet fase/maksimum fase).
- lengden på impulsresponsen (FIR/IIR).
- type filter (notch/ kam/ digital resonator/ allpassfilter/ lavpass/ høypass/ båndpass/ båndstopp)

Merknad 1! Alle svar skal begrunnes. Tilfeldige kombinasjoner uten forklaring belønnes ikke.
Merknad 2! Ved å bruke hodet og det dere har lært i kurset, er det ikke sikkert dere trenger å beregne noe som helst i denne oppgaven!
Merknad 3! Det gis 0.33 poeng for hver riktig ligning-figur kombinasjon og 0.10 poeng for hver riktig tilleggsopplysning. Oppgaven gir maksimalt 5 poeng.

(Fortsettes på side 9.)

$$y[n] = 0.9\sqrt{2}y[n-1] - 0.81y[n-2] + x[n] \quad (11)$$

$$H(z) = \frac{1}{2}\sum_{k=0}^{M-1}(W_M^{-mk} + W_M^{mk})z^{-k}, M = 10, m = 1, W_m = e^{j2\pi/M} \quad (12)$$

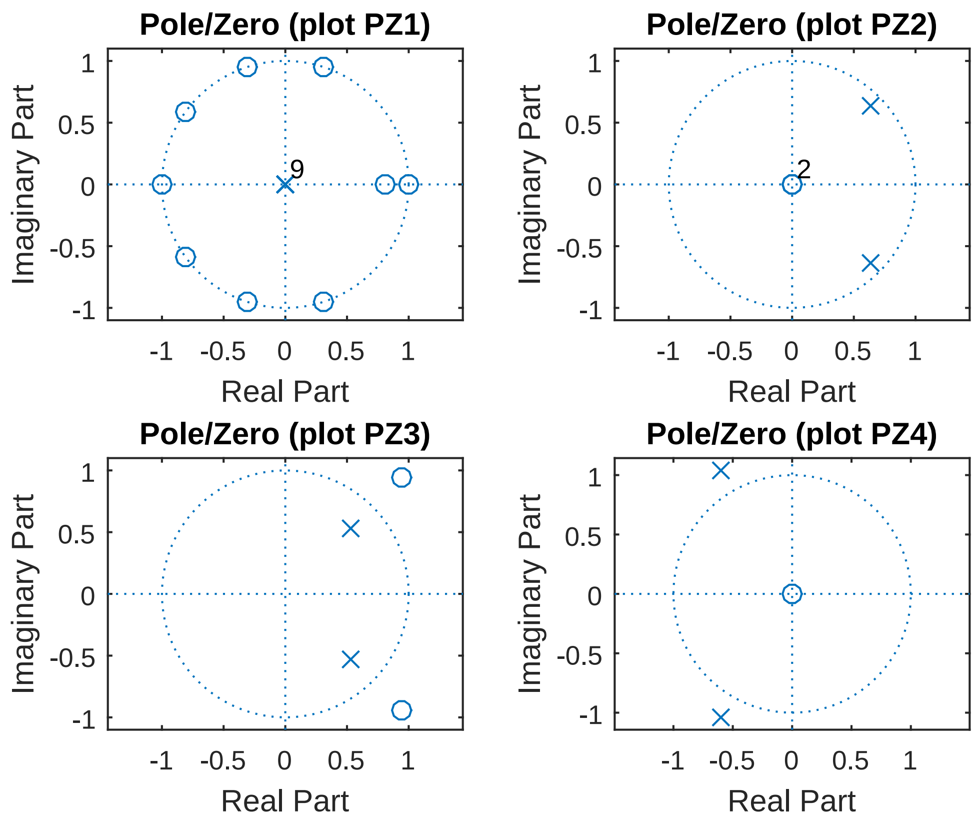
$$H(z) = \frac{(3/4)(3/4 - \sqrt{2}z^{-1}) + z^{-2}}{1 - (3\sqrt{2}/4)z^{-1} + (3/4)^2 z^{-2}} \quad (13)$$

$$H(z) = \frac{z^{-1}}{1 + (6/5)z^{-1} + (6/5)^2 z^{-2}} \quad (14)$$

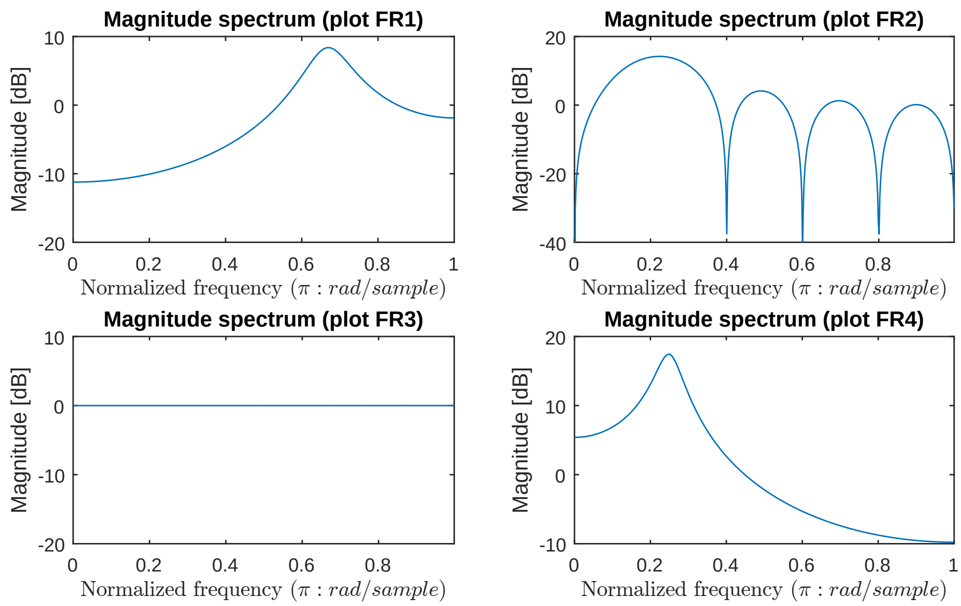
Fyll ut en tabell på følgende format (Begrunnelser og beregninger tar du utenfor tabellen):

Filter → Karakteristikk ↓	Ligning 11	Ligning 12	Ligning 13	Ligning 14
Pol/Nullpkt (figur)				
Magnituderespons (figur)				
Impulsrespons (figur)				
Stabilitetsegenskaper				
Faserespons				
Impulsrespons lengde				
Type				

(Fortsettes på side 10.)

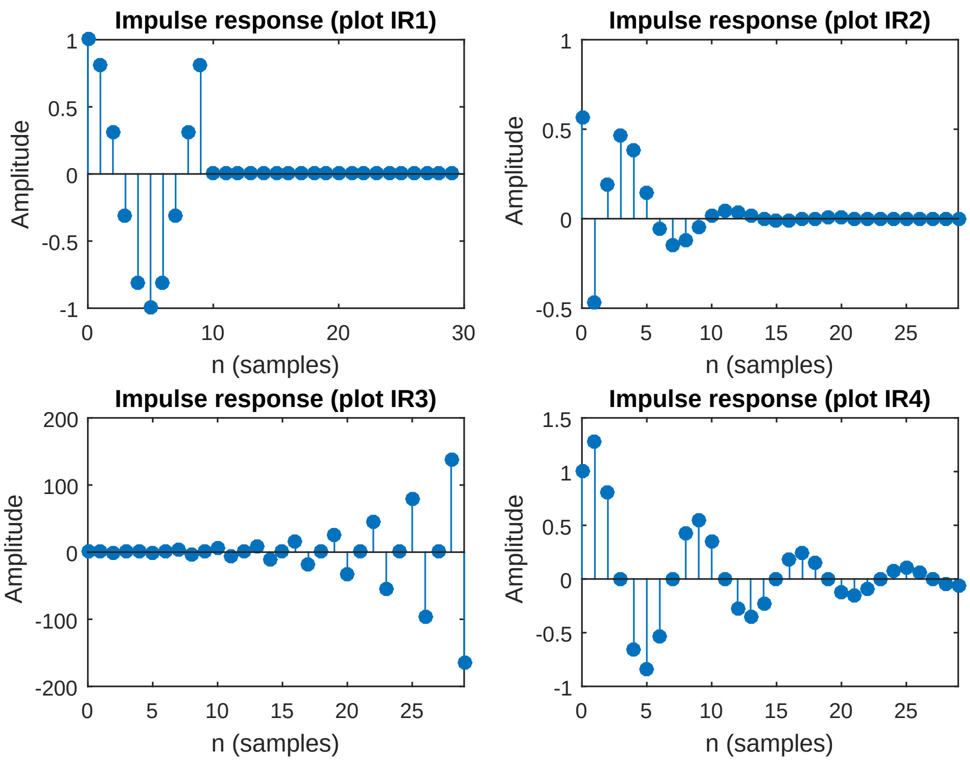


Figur 1: Pole/Zero plots



Figur 2: Magnitude spectra

(Fortsettes på side 11.)



Figur 3: Impulse responses

Svar:

- System 11: Plots PZ2, FR4 and IR4. Stable. Minimum phase. IIR. Digital resonator (optionally also Lowpass).
- System 12: Plots PZ1, FR2 and IR1. Stable. Minimum phase. FIR. Bandpass ("moving average type")
- System 13: Plots PZ3, FR3 and IR2. Stable. Maximum phase. IIR. Allpass
- System 14: Plots PZ4, FR1 and IR3. Unstable. IIR (optionally Highpass).

We can do the following reasoning to get this result. Let us consider equation 12 first. This is the only one having 9 zeros, and we can immediately recognize it as PZ1. Using our knowledge about effect of poles/zeros on frequency response, we can also identify it as FR2 as zeros pull the magnitude spectrum down. Since it has no poles outside origin, it is a FIR filter, meaning the impulse response must be IR1. FIR filters are always stable. The distribution of zeros suggests it is a bandpass filter centered approximately at pi/4. All zeros are on or inside the unit circle, so the filter is considered minimum phase.

(Fortsettes på side 12.)

Second, we consider equation 13. We note that the numerator and denominator are reversed copies of each other, suggesting we are talking about an allpass filter. We recognize PZ3 as the allpass filter (or as the only plot with both zeros and poles outside the origin). The flat spectrum in FR3 is also the response of an allpass filter. All poles are inside the unit circle, so it is stable. Since it has poles not at the origin, it is an IIR filter. All zeros are outside the unit circle, so it is a maximum phase filter. By exclusion, we can say the the only possible impulse responses are IR2 or perhaps IR4. We will come back to the correct choice.

Third, let us consider equation 11. We can see from the impulse response that it has no zeros and two poles (second order equation in z^{-1}). At first glance, it seems we cannot differ between PZ2 and PZ4. But we note that there are two zeros in the origin at PZ2 (to get N=M), which fits well with this system. This is a bit "sketchy" reasoning, and it would probably be a safer solution to solve for the poles. But after having decided on PZ2, we can conclude that it is stable. The poles at approximately $\pm\pi/4$, causes the spike in FR4. The impulse response is either IR2 or IR4. Comparing PZ2 and PZ3, we see that the poles at PZ2 is closer to the unit circle, meaning that the response will be less dampened than for PZ3. This lets us conclude that IR2 belongs to PZ3 or the system in equation 13. IR4 belongs to PZ2, and thus, the system in equation 11. Another option would be to calculate the first three values of $y[n]$ for $x[n] = \delta[n]$. It will then be obvious that IR4 is the correct impulse response. The system has no zeros outside the unit circle and is a minimum phase IIR filter. Based on the pole location and impulse response, we recognize the filter as a digital resonator, but it can also be considered a filter having a lowpass character.

The fourth system is equation 14. This has two poles and one zero at ∞. PZ4 has two poles and one added zero at the origin to get N=M, which fits well with this system. We see that this is an unstable system having two poles outside the unit circle. The poles look to be at an angle $3\pi/4$, so the magnitude response should be FR1. Since it is an unstable IIR filter, IR3 is the only candidate for the impulse response. It is of limited value to discuss properties of an unstable filter, but based on the magnitude response, it can be termed a highpass filter.

Oppgave 5 Lineær vs. sirkulær konvolusjon

Vi studerer to sekvenser med endelig lengde

$$x_1[n] = \{\underset{\uparrow}{1}, -2, 1, -3\}$$

$$x_2[n] = \{\underset{\uparrow}{0}, 2, -1, 0, 0, 4\}$$

a) Beregn den lineære konvolusjonen $x_1[n] * x_2[n]$. 1 p.

b) Beregn den 6-punkt sirkulære konvolusjonen $x_1[n] \circledast_6 x_2[n]$. 1 p.

(Fortsettes på side 13.)

c) Hva bør den minste verdien for N være for at den N-punkt sirkulære konvolusjonen skal være lik den lineære konvolusjonen? 0.5 p.

d)
- Når en filtreringsoperasjon beregnes på en datamaskin gjøres det som regel ved å bruke multiplikasjon i frekvensdomenet. Hva slags konvolusjon tilsvarer det? 0.5 p.
- Forklar hvorfor null-padding er en nyttig teknikk i dette tilfellet. 0.5 p.

Svar:

a) We recall the formula for the linear convolution:

$$h[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$

Therefore

$$\begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \\ h[4] \\ h[5] \\ h[6] \\ h[7] \\ h[8] \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -5 \\ 4 \\ -7 \\ 7 \\ -8 \\ 4 \\ -12 \end{bmatrix}$$

b) Likewise, the formula for the N-point circular convolution is

$$g[n] = x_1[n]\, \circledN\, x_2[n] = \sum_{k=0}^{N-1} x_1[k]x_2[< n-k >_N]$$

For computing a 6-point circular convolution, we will need $x_2[n]$ and the zero-padded version of $x_1[n]$, $x_{1zp}[n]$:

$$x_{1zp}[n] = \{\underset{\uparrow}{1}, -2, 1, -3, 0, 0\}$$

Therefore

$$\begin{bmatrix} g[0] \\ g[1] \\ g[2] \\ g[3] \\ g[4] \\ g[5] \end{bmatrix} \begin{bmatrix} 0 & 4 & 0 & 0 & -1 & 2 \\ 2 & 0 & 4 & 0 & 0 & -1 \\ -1 & 2 & 0 & 4 & 0 & 0 \\ 0 & -1 & 2 & 0 & 4 & 0 \\ 0 & 0 & -1 & 2 & 0 & 4 \\ 4 & 0 & 0 & -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -8 \\ 6 \\ -17 \\ 4 \\ -7 \\ 7 \end{bmatrix}$$

(Fortsettes på side 14.)

c) For the circular convolution to be equal to the linear convolution, it has to have a length of at least $N_1 + N_2 - 1$ where N_1 and N_2 are the length of x_2 and x_1, respectively. This means that the smallest value of N so that the N-point circular convolution is equal to the linear convolution is $N = 9$. We can verify this by computing the 9-point circular convolution:

$$\begin{bmatrix} g[0] \\ g[1] \\ g[2] \\ g[3] \\ g[4] \\ g[5] \\ g[6] \\ g[7] \\ g[8] \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 4 & 0 & 0 & -1 & 2 \\ 2 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & -1 \\ -1 & 2 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 4 \\ 4 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -5 \\ 4 \\ -7 \\ 7 \\ -8 \\ 4 \\ -12 \end{bmatrix}$$

which gives the result obtained in a).

d)
- Using a computer for a filtering operation implies the use of the DFT and IDFT (actually FFT and IFFT). The product of the DFTs corresponds to the DFT of the circular convolution.
- In filtering operations and more generally in design of LTI systems, we are interested in the linear convolution between the input and the system impulse response. To make sure we get the desired result when using a computer and the DFT/IDFT (FFT/IFFT), we need to ensure the N-point circular convolution gives the same result as a linear convolution. To do that we use zero-padding and pad the input sequences to a length N with $N \geq N_1 + N_2 - 1$ where N_1 and N_2 are the length of each sequence.

Formelsamling

Grunnleggende sammenhenger:

$$\begin{aligned}
\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\
\cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\
\sin 2\alpha &= 2\sin\alpha\cos\alpha \\
\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\
\sin\alpha + \sin\beta &= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\sin\alpha - \sin\beta &= 2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\
\cos\alpha + \cos\beta &= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\cos\alpha - \cos\beta &= -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\
\cos^2\alpha + \sin^2\alpha &= 1 \\
\cos\alpha &= \frac{1}{2}(e^{j\alpha} + e^{-j\alpha}) \\
\sin\alpha &= \frac{1}{2j}(e^{j\alpha} - e^{-j\alpha}) \\
\sum_{n=0}^{N-1} a^n &= \begin{cases} N & \text{for } a = 1 \\ \frac{1-a^N}{1-a} & \text{ellers} \end{cases} \\
ax^2 + bx + c = 0 &\Leftrightarrow x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

Lineær konvolusjon:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] = h[n] * x[n]$$

Sirkulær konvolusjon:

$$y[n] = x[n] \circledN h[n] = \sum_{k=0}^{N-1} x[k]h[< n-k >_N] = \sum_{k=0}^{N-1} x[< n-k >_N]h[k] = h[n] \circledN x[n]$$

Diskret tid-fouriertransformasjon (DTFT):

$$\begin{aligned}
\text{Analyse:} \quad X(\Omega) &= = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n} \\
\text{Syntese:} \quad x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega
\end{aligned}$$

Diskret fouriertransformasjon (DFT):

$$\begin{aligned}
\text{Analyse:} \quad X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}, \quad 0 \le k \le N-1 \\
\text{Syntese:} \quad x[n] &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N}, \quad 0 \le k \le N-1
\end{aligned}$$

(Fortsettes på side 16.)

z-transformasjonen:

$$\text{Analyse:} \qquad X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$