UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in: Fasit for INF3470/4470 — Digital signal processing

Day of examination: December 2007, v01

This problem set consists of 5 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

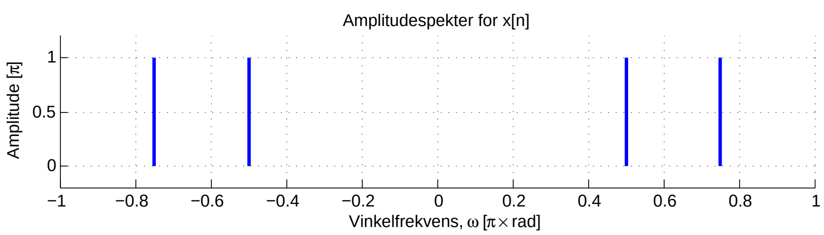
Problem 1 Sampling og rekonstruksjon

a)

1. $f_s > 2 * 300\text{Hz} = 600\text{Hz}$.
2. $f_s > 2 * 300\text{Hz} = 600\text{Hz}$.
3. $f_s > 2 * (151/2)\text{Hz} = 151\text{Hz}$.
4. ikke mulig ('$f_s > 2 \times \infty$').

b)

$$
\begin{aligned}
x[n] &= x(nT_s) = x(n/f_s) = \cos(300\pi * n/400 + \pi/2) + \cos(600\pi * n/400 + \pi/3) \\
&= \cos(\frac{3}{4}\pi n + \pi/2) + \cos(\frac{6}{4}\pi n + \pi/3) \\
&= \cos(\frac{3}{4}\pi n + \pi/2) + \cos(-\frac{1}{2}\pi n + \pi/3) \\
&= \cos(\frac{3}{4}\pi n + \pi/2) + \cos(\frac{1}{2}\pi n - \pi/3)
\end{aligned}
$$



Plott av amplitudespekter for $x[n]$.

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Problem 2 LTI-systemer

a) F.eks: Stabilt hvis $\sum_n |h[n]| < \infty$.

b) F.eks: System stabilt hvis $h[n] = 0$ for $n < 0$.

c) Lineært, dvs. additivt og homogent:
$\mathrm{T}\{a_1 x_1[n] + a_2 x_2[n]\} = a_1 \mathrm{T}\{x_1[n]\} + a_2 \mathrm{T}\{x_2[n]\}$.

Tidsinvariant: A linear system T is *time-invariant* or *shift-invariant* iff the following is true:

$$x(n) \longrightarrow \boxed{\mathrm{T}\{\cdot\}} \longrightarrow \quad y[n] \longrightarrow \boxed{\text{Shift by } k} \longrightarrow y[n-k]$$
$$x(n) \longrightarrow \boxed{\text{Shift by } k} \longrightarrow x[n-k] \longrightarrow \boxed{\mathrm{T}\{\cdot\}} \longrightarrow \quad y[n-k].$$

d)

1. **Linearitet:** La

$$\begin{aligned} y_1[n] &= x_1[n] - 3x_1[n-1] + x_1[n-2], \\ y_2[n] &= x_2[n] - 3x_2[n-1] + x_2[n-2] \\ \text{og } x[n] &= \alpha x_1[n] + \beta x_2[n]. \end{aligned}$$

Har da at

$$\begin{aligned} y[n] &= x[n] - 3x[n-1] + x[n-2] \\ &= \alpha x_1[n] + \beta x_2[n] - 3 * (\alpha x_1[n-1] + \beta x_2[n-1]) \\ &\quad + \alpha x_1[n-2] + \beta x_2[n-2] \\ &= \alpha y_1[n] + \beta y_2[n]. \end{aligned}$$

Systemet er derfor lineært.

Tidsinvarians: La $v[n] = x[n - n_0]$. Har da at

$$\begin{aligned} w[n] &= v[n] - 3v[n-1] + v[n-2] \\ &= x[n-n_0] - 3x[n-1-n_0] + x[n-2-n_0] \\ &= x[(n-n_0)] - 3x[(n-n_0)-1] + x[(n-n_0)-2], \\ \text{og } y[n-n_0] &= x[(n-n_0)] - 3x[(n-n_0)-1] + x[(n-n_0)-2]. \end{aligned}$$

Siden $w[n]$ er identisk med $y[n - n_0]$ er systemet tidsinvariant.

2. **Linearitet:** La

$$\begin{aligned} y_1[n] &= x_1[n+1] - x_1[n] + x_1[n-1] + 5x_1[2], \\ y_2[n] &= x_2[n+1] - x_2[n] + x_2[n-1] + 5x_2[2], \\ \text{og } x[n] &= \alpha x_1[n] + \beta x_2[n]. \end{aligned}$$

Har da at

$$\begin{aligned} y[n] &= x[n+1] - x[n] + x[n-1] + 5x[2] \\ &= \alpha x_1[n+1] + \beta x_2[n+1] - (\alpha x_1[n] + \beta x_2[n]) \\ &\quad + \alpha x_1[n-1] + \beta x_2[n-1] + \alpha x_1[2] + \beta x_2[2] \\ &= \alpha y_1[n] + \beta y_2[n]. \end{aligned}$$

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Systemet er derfor lineært.

Tidsinvarians: La $v[n] = x[n - n_0]$. Har da at

$$\begin{aligned} w[n] &= v[n+1] + v[n] + v[n-1] + 5v[2] \\ &= x[n+1-n_0] - x[n-n_0] + x[n-1-n_0] + 5x[2-n_0] \\ &= x[(n-n_0)+1] - x[(n-n_0)] + x[(n-n_0)-1] + 5x[2-n_0], \\ \text{og } y[n-n_0] &= x[(n-n_0)+1] - x[(n-n_0)] + x[(n-n_0)-1] + 5x[2]. \end{aligned}$$

Siden $w[n]$ er ikke identisk med $y[n - n_0]$ er systemet ikke tidsinvariant (evt. tidsvariant/tidsvarierende).

Problem 3 Design av IIR filtre

a) $r = 1 - (20/1000)\pi = 1 - 6.28/100 = 1 - 0.0628 = 0.9372$

$$\begin{aligned} H(z) &= \frac{(z-1)(z+1)}{(z - re^{jpi/2})(z - re^{-jpi/2})} \\ &= \frac{z^2 - 1}{z^2 + r^2} = \frac{1 - z^{-2}}{1 + r^2 z^{-2}}, \\ & r^2 = 0.877969 \end{aligned}$$

b) Fra $H(z) = Y(z)/X(z)$ fås $Y(z)(1 + r^2 z^{-2}) = X(z)(1 - z^{-2})$
Fra avlesning fås diff.eq: $y[n] = -r^2 y[n-2] + x[n] - x[n-2]$.

Problem 4 Design av FIR-filtre

a) Minste antall koeffisienter: 5. Trenger 2+2 nullpkt, noe som gir en systemfunksjon av 4-orden, dvs. 5 koeffisienter.

b) Nullpkt: $z_1 = e^{j0.5\pi}$, $z_2 = z_1^*$, $z_3 = e^{j0.75\pi}$ og $z_4 = z_3^*$.

c)

$$\begin{aligned} H(z) &= (a/z^4)(z - z_1)(z - z_2)(z - z_3)(z - z_4) \\ &= (a/z^4)(z^2 - z(z_1 + z_1^*) + z_1 z_1^*)(z^2 - z(z_3 + z_3^*) + z_3 z_3^*). \end{aligned}$$

Har at $z_1 z_1^* = z_3 z_3^* = 1$, $z_1 + z_1^* = 2\text{Re}(z_1) = 0$ og $z_3 + z_3^* = 2\text{Re}(z_3) = -\sqrt{2}$. Får da at

$$\begin{aligned} H(z) &= (a/z^4)(z^2 + 1)(z^2 + \sqrt{2}z + 1) \\ &= (a/z^4)(z^4 + \sqrt{2}z^3 + 2z^2 + \sqrt{2}z + 1) \\ &= a(1 + \sqrt{2}z^{-1} + 2z^{-2} + \sqrt{2}z^{-3} + z^{-4}). \end{aligned}$$

Ved avlesning fås: $h[n] = a\{\underset{\uparrow}{1}, \sqrt{2}, 2, \sqrt{2}, 1\}$.

Plott av poler og nullpkt:

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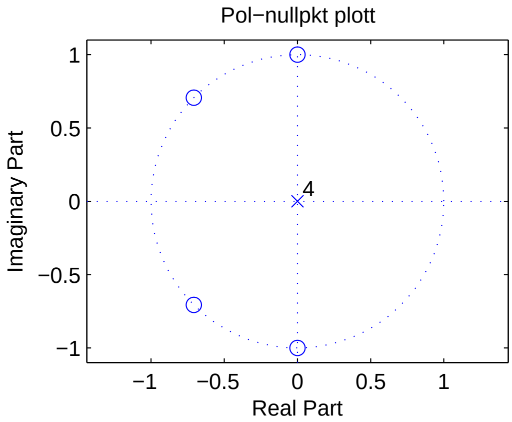

Plott av pol-nullpunktsplott.

d)

$$\begin{aligned} H(e^0) &= a(1+\sqrt{2}+2+\sqrt{2}+1) = 1 \\ a(4-2\sqrt{2}) &= 1 \\ a &= 1/(4+2\sqrt{2}) \\ &\Rightarrow \quad h[n] = \frac{1}{4+2\sqrt{2}}\{\underset{\uparrow}{1},\ \sqrt{2},\ 2,\ \sqrt{2},\ 1\} \end{aligned}$$

Problem 5 Filtre og konvolusjon

a) $y[n] = h[n] * x[n] = \sum_{k=0}^{2} h[k]x[n-k] = x[n] - 2x[n-1] + x[n-2]$.

b) $y[n] = \{\underset{\uparrow}{1},\ -2,\ 4,\ -6,\ 5,\ -4,\ 2\}$

c)

$$\begin{aligned} Y(e^{\jmath\hat{\omega}}) &= \sum_{n=-\infty}^{\infty} y[n]e^{-\jmath\hat{\omega}n} = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h[k]x[n-k]e^{-\jmath\hat{\omega}n} \\ &= \sum_{k=-\infty}^{\infty} h[k]\left(\sum_{n=-\infty}^{\infty} x[n-k]e^{-\jmath\hat{\omega}n}\right) = \sum_{k=-\infty}^{\infty} h[k]\left(\sum_{m=-\infty}^{\infty} x[m]e^{-\jmath\hat{\omega}(m+k)}\right) \\ &= \sum_{k=-\infty}^{\infty} h[k]\left(\sum_{m=-\infty}^{\infty} x[m]e^{-\jmath\hat{\omega}m}\right)e^{-\jmath\hat{\omega}k} = X(e^{\jmath\hat{\omega}})\sum_{k=-\infty}^{\infty} h[k]e^{-\jmath\hat{\omega}k} \\ &= X(e^{\jmath\hat{\omega}})H(e^{\jmath\hat{\omega}}) \qquad \text{q.e.d} \end{aligned}$$

d)

$$\begin{aligned} H(e^{\jmath\hat{\omega}}) &= H(z)|_{z=e^{\jmath\hat{\omega}}} = \frac{1}{M}\frac{e^{\jmath\hat{\omega}M}-1}{e^{\jmath\hat{\omega}(M-1)}(e^{\jmath\hat{\omega}}-1)} \\ &= \frac{1}{M}\frac{(e^{\jmath\hat{\omega}M/2}-e^{-\jmath\hat{\omega}M/2})e^{\jmath\hat{\omega}M/2}}{e^{\jmath\hat{\omega}(M-1)}e^{\jmath\hat{\omega}/2}(e^{\jmath\hat{\omega}/2}-e^{-\jmath\hat{\omega}/2})} \\ &= \frac{1}{M}\frac{\sin(M\hat{\omega}/2)}{e^{\jmath\hat{\omega}(M-1)/2}\sin(\hat{\omega}/2)} \end{aligned}$$

(Continued on page 5.)

GM filteret som def i d) glatter signalet ved at utgangssignalet blir et lokalt midlet utgave av inngangssignalet. Dvs at GM er et lavpassfilter; det fjerner høyfrekvente/raskt varierende deler av et signal.

e)

$$\begin{aligned}
h_K[n] &= h[n] * h[n] * \cdots * h[n] \\
\Rightarrow\ H_K(z) &= H(z) \cdot H(z) \cdots H(z) = H^K(z) \\
\Rightarrow\ H_K(e^{\jmath\hat{\omega}}) &= H_K(z)|_{z=e^{\jmath\hat{\omega}}} = \frac{1}{M^K} \frac{\sin^K(M\hat{\omega}/2)}{e^{\jmath\hat{\omega}K(M-1)/2} \sin^K(\hat{\omega}/2)}
\end{aligned}$$