UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in	INF3470/4470 — Digital signal processing
Day of examination:	December 9th, 2011
Examination hours:	14.30 – 18.30

This problem set consists of 13 pages.

Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Answer:

Suggested solutions, 22/12-2011:

Problem 1 z-transformation

a) Find the z-transform $X(z)$ of the expression $x[n] = \alpha^n u[n] - \alpha^n u[n-5]$.

1 p.

b) Find the causal sequence $h[n]$ with z-transform $H(z) = \frac{z+3}{z-2}$. 1 p.

c) A signal $x[n]$ has the z-transform $X(z)$: Show that multiplying $x[n]$ with n corresponds to multiplying $X(z)$ with $-z\frac{d}{dz}$: 2 p.

$$n\,x[n] \overset{Z}{\longleftrightarrow} -z\frac{d}{dz}X(z)$$

How does this affect the ROC?

(Continued on page 2.)

Answer:

a) We compute $X(z)$ by taking the z-transform of $x[n]$:

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \\
&= \sum_{n=-\infty}^{\infty} \alpha^n \left(u[n] - u[n-5]\right) z^{-n} \\
&= \sum_{n=0}^{4} \alpha^n\, z^{-n} \\
&= 1 + \alpha\, z^{-1} + \alpha^2\, z^{-2} + \alpha^3\, z^{-3} + \alpha^4\, z^{-4} \\
&= \frac{1 - \alpha^5\, z^{-5}}{1 - \alpha\, z^{-1}}.
\end{aligned}$$

This sequence is finite and right-sided, hence the ROC is the entire z-plane except in $z = 0$.

b) Let's start by expanding $H(z)$ slightly:

$$H(z) = \frac{z+3}{z-2} = \frac{1 + 3\, z^{-1}}{1 - 2\, z^{-1}} = \frac{1}{1 - 2\, z^{-1}} + \frac{3\, z^{-1}}{1 - 2\, z^{-1}}$$

Both the causal sequence $h[n] = \alpha^n\, u[n]$ and the anti-causal sequence $h[n] = -\alpha^n\, u[-n-1]$ has the z-transform $H(z) = \frac{1}{1-\alpha\, z^{-1}}$. Since we're speaking of the former case here, finding the inverse transform is simply a matter of recognizing the terms and remembering that a multiplication by z^{-1} in the z-domain corresponds to a $n-1$ shift in the time domain:

$$h[n] = 2^n\, u[n] + 3\, 2^{n-1}\, u[n-1]$$

Alternative:
We find the difference equation:

$$\begin{aligned}
H(z) = \frac{z+3}{z-2} &= \frac{1 + 3\, z^{-1}}{1 - 2\, z^{-1}} = \frac{Y(z)}{X(z)} \\
Y(z)(1 - 2\, z^{-1}) &= X(z)(1 + 3\, z^{-1}) \\
&\Downarrow \text{IZT} \\
y[n] &= 2\, y[n-1] + x[n] + 3\, x[n-1]
\end{aligned}$$

(Continued on page 3.)

Then apply an impulse $x[n] = \delta[n]$:

$$\begin{aligned} y[0] &= \delta[0] = 1 \\ y[1] &= 2\,y[0] + 3\,\delta[1-1] = 5 \\ y[2] &= 2\,y[1] = 10 \\ y[2] &= 2\,y[2] = 20 \\ y[2] &= 2\,y[2] = 40 \\ &\vdots \end{aligned}$$

From this we may infer the symbolic expression of the impulse response:

$$\begin{aligned} h[n] &= 5\,2^{n-1}\,u[n-1] + \delta[n] \\ &= 2^n\,u[n] + 3\,2^{n-1}\,u[n-1] \end{aligned}$$

c) One may start in either "end" of this proof, any method is equally valid.

$$\begin{aligned} \frac{dX(z)}{dz} &= \frac{d}{dz} \sum_{n=-\infty}^{\infty} x[n]\,z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \frac{d}{dz} x[n]\,z^{-n} \\ &= \sum_{n=-\infty}^{\infty} (-n)\,x[n]\,z^{-n-1} \qquad \Big|\cdot -z \\ -z\frac{dX(z)}{dz} &= \sum_{n=-\infty}^{\infty} n\,x[n]\,z^{-n} \qquad \text{Q.E.D.} \end{aligned}$$

What happens with the ROC? A "proof" was not requested, but the logic is quite simple: If we let $X(z) = \frac{B(z)}{A(z)}$, then it follows by the quotient rule that $-z\frac{d}{dz}X(z) = -z\,X'(z) = -z\frac{A'(z)B(z)-A(z)B'(z)}{A^2(z)}$. That is, the poles are duplicated, and the ROC is unaffected (a sufficient answer).

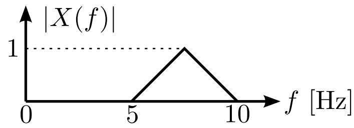
(Continued on page 4.)

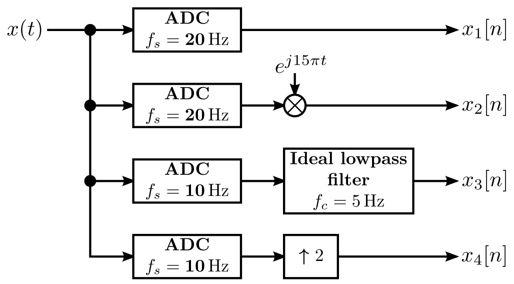
Problem 2 Sampling

a) Explain briefly the meaning of the following terms: 2 p.

- Sampling frequency and Nyquist frequency
- Quantization and quantization error
- Uniform and non-uniform sampling
- Frequency aliasing

b) A real signal $x(t)$ with the magnitude response $|X(f)|$ (as shown to the right) is sampled and filtered in four different ways:





ADC	Analog-to-Digital Converter
f_s	Sample frequency
f_c	Filter cut-frequency
$\uparrow$	Upsampling by zero-expansion between samples

Sketch the magnitude responses $|X_1(f)|$, $|X_2(f)|$, $|X_3(f)|$ and $|X_4(f)|$ in the frequency range -10 Hz to 10 Hz. Assume that the amplitude of the signal is unaffected by the upsampling process. 2 p.

Answer:

a)

- **Sampling frequency** is the rate at which we sample an underlying signal. To avoid frequency aliasing, the (Nyquist-Shannon) sampling theorem states that the sample rate must be twice the highest frequency in the signal. Or, in case the signal is bandlimited, the sampling rate must be twice the bandwidth of that signal. This sampling rate is called **Nyquist rate**, and the **Nyquist frequency** is (usually) half of that. You'll get credit for either answer, as the distinction has not been made explicit.
- **Quantization** is the procedure of constraining the signal value at the sample point to a discret value. By doing so the sample values are not the true ones, only an approximation, and the error involved here is referred to as **quantization error**.
- **Uniform sampling** is when the sample rate is static, and nonuniform sampling is when the sampling rate is dynamic.

(Continued on page 5.)

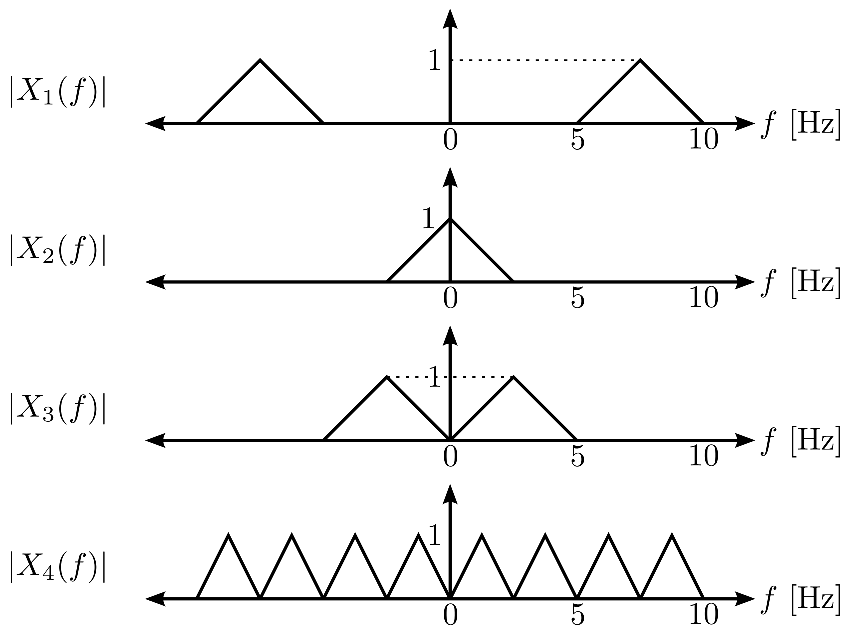
- **Frequency aliasing**, or just alias, is when frequencies appear in the sampled signals spectrum that was not in the original signals spectrum. This is an effect of undersampling.

b) **(1)** The signal is sampled at twice the highest frequency, and no aliasing occurs. Since the signal is real, the spectrum will be symmetric.

(2) The signal is sampled as in (1), but is multiplied with a complex exponential with the phase $\Theta = 15\pi t = 2\pi 7.5t$, and hence frequency 7.5 Hz. Hence, this corresponds to shifting the frequency spectra by 7.5 Hz.

(3) The signal is sampled at the highest frequency, but at twice the bandwidth. Aliasing occurs, but no harm is done since the frequency band is simply duplicated onto lower frequencies where there was no signal before. By applying a lowpass filter, only the low frequency duplicate of the band with remain.

(4) The signal is undersampled as in (3), and then upsampled by means of intersample zero-padding. This result is a spectrum compressed by a factor equal to the upsampling factor (i.e. 2 here).



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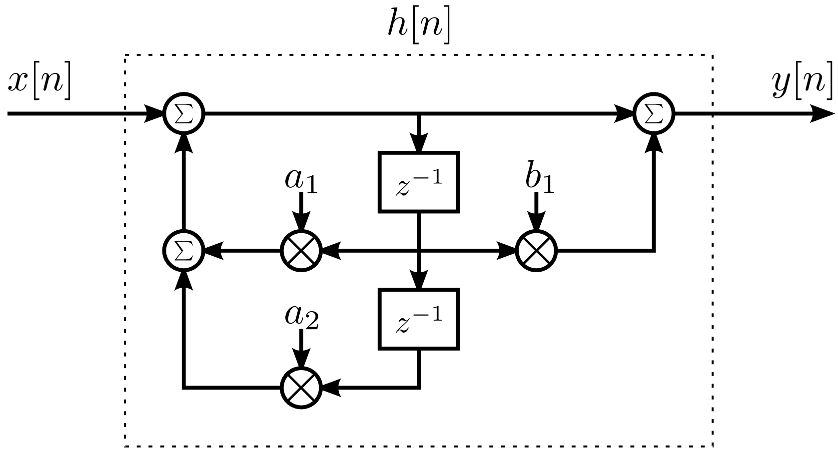
Problem 3 System analysis

A filter is described as:



where ⓢ indicates sum and ⊗ indicates multiplication.

a) Find the system function $H(z)$. 1 p.

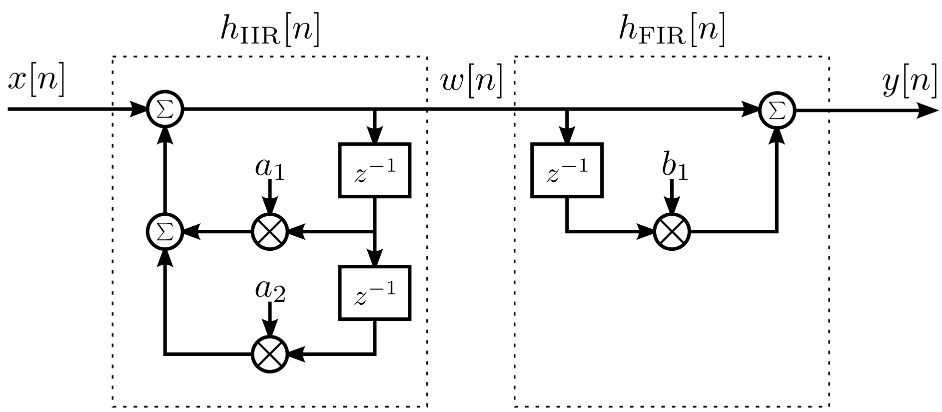
In the following two exercises, assume that $H(z) = \frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$:

b) Plot poles and zeros, and determine whether this filter is stable and/or causal. 2 p.

c) Find the expressions for the filter magnitude response $|H(\Omega)|$ and phase response $\Theta_H(\Omega)$. The final expressions should not contain complex numbers. Avoid spending time trying to simplify the terms. 2 p.

Answer:

a) The easiest way to find $H(z)$ is to first decompose this system into its FIR and IIR part:



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Then we set up the difference equation for each of the two systems:

$$\begin{aligned} w[n] &= a_1\, w[n-1] + a_2\, w[n-2] + x[n] \\ y[n] &= w[n] + b_1\, w[n-1] \end{aligned}$$

Take the z-transform:

$$\begin{aligned} W(z) &= a_1\, W(z)\, z^{-1} + a_2\, W(z)\, z^{-2} + X(z) \\ Y(z) &= W(z) + b_1\, W(z)\, z^{-1} \end{aligned}$$

Then rearrange the expressions to arrive at the FIR and IIR transferfunctions:

$$\begin{aligned} H_{\mathrm{IIR}}(z) &= \frac{W(z)}{X(z)} = \frac{1}{1 - a_1\, z^{-1} - a_2\, z^{-2}} \\ H_{\mathrm{FIR}}(z) &= \frac{Y(z)}{W(z)} = 1 + b_1\, z^{-1} \end{aligned}$$

Since the two systems are put in series, $H(z)$ is found by simply multiplying $H_{\mathrm{IIR}}(z)$ and $H_{\mathrm{FIR}}(z)$ together:

$$H(z) = H_{\mathrm{IIR}}(z)\, H_{\mathrm{FIR}}(z) = \frac{1 + b_1\, z^{-1}}{1 - a_1\, z^{-1} - a_2\, z^{-2}}$$

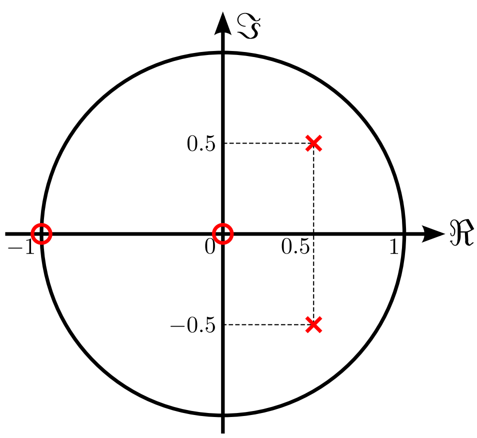
b) To find the zeros and poles we get rid of the negative exponentials of z and factorize $H(z)$:

$$H(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5 z^{-2}}\, \frac{z^2}{z^2} = \frac{z^2 + z}{z^2 - z + 0.5} = \frac{z(z+1)}{(z - p_1)(z - p_2)}$$

where $\{p_1, p_2\} = 0.5 \pm j0.5$. This is easily found by from the equation of second degree polynomials. The zeros are recognized directly: $\{z_1, z_2\} = \{0, -1\}$.

The poles and zeros are plotted to the right.

Whether $H(z)$ is stable or causal is a trick question, as one must be known in order to know the other. If the system is causal, all the poles must lie within the unit circle to ensure stability, and if it is anti-causal all the poles must lie on the outside. Hence, assuming causality, the system is stable, otherwise not.



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c) The first step is to evaluate the transferfunction along the unit circle, i.e. setting $z = e^{j\Omega}$:

$$
\begin{aligned}
H(\Omega) &= \frac{z^2+z}{z^2-z+0.5}\Big|_{z=e^{j\Omega}} \\
&= \frac{e^{j2\Omega}+e^{j\Omega}}{e^{j2\Omega}-e^{j\Omega}+0.5}\,\frac{e^{-j\frac{3\Omega}{2}}}{e^{-j\frac{3\Omega}{2}}} \\
&= \frac{e^{j\frac{\Omega}{2}}+e^{-j\frac{\Omega}{2}}}{e^{j\frac{\Omega}{2}}-e^{-j\frac{\Omega}{2}}+0.5e^{-j\frac{3\Omega}{2}}} \\
&= \frac{2\cos\frac{\Omega}{2}}{2j\sin\frac{\Omega}{2}+0.5(\cos\frac{3\Omega}{2}+j\sin\frac{3\Omega}{2})}
\end{aligned}
$$

Magnitude response:
Using the expression above:

$$
\begin{aligned}
\left|H(\Omega)\right| &= \left|\frac{2\cos\frac{\Omega}{2}}{2j\sin\frac{\Omega}{2}+0.5(\cos\frac{3\Omega}{2}+j\sin\frac{3\Omega}{2})}\right| \\
&= \frac{\left|2\cos\frac{\Omega}{2}\right|}{\left|2j\sin\frac{\Omega}{2}+0.5(\cos\frac{3\Omega}{2}+j\sin\frac{3\Omega}{2})\right|} \\
&= \frac{2\cos\frac{\Omega}{2}}{\sqrt{(0.5\cos\frac{3\Omega}{2})^2+(2\sin\frac{\Omega}{2}+0.5\sin\frac{3\Omega}{2})^2}}
\end{aligned}
$$

Alternative (just as valid):

$$
\begin{aligned}
\left|H(\Omega)\right|^2 &= H(\Omega)\,H^*(\Omega) \\
&= \frac{e^{j2\Omega}+e^{j\Omega}}{e^{j2\Omega}-e^{j\Omega}+0.5}\,\frac{e^{-j2\Omega}+e^{-j\Omega}}{e^{-j2\Omega}-e^{-j\Omega}+0.5} \\
&= \frac{2+2\cos\Omega}{2.25+3\cos\Omega+cos2\Omega}
\end{aligned}
$$

Phase response:

$$
\begin{aligned}
\Theta_H(\Omega) &= \angle\left[\frac{2\cos\frac{\Omega}{2}}{2j\sin\frac{\Omega}{2}+0.5(\cos\frac{3\Omega}{2}+j\sin\frac{3\Omega}{2})}\right] \\
&= \angle\left[2\cos\frac{\Omega}{2}\right]-\angle\left[2j\sin\frac{\Omega}{2}+0.5(\cos\frac{3\Omega}{2}+j\sin\frac{3\Omega}{2})\right] \\
&= -\arctan\frac{2\sin\frac{\Omega}{2}+0.5\sin\frac{3\Omega}{2}}{0.5\cos\frac{3\Omega}{2}}
\end{aligned}
$$

This expression is not the easiest to evaluate, so you'll get credit if you've shown the concept of arriving at the magnitude and phase response.

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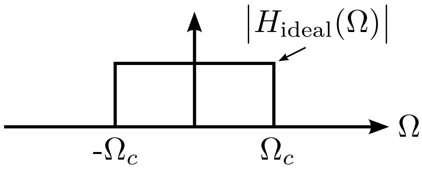
Problem 4 FIR filter design

The magnitude response of an ideal lowpass filter is given as:



a) Find the impulse response $h_{\text{ideal}}[n]$ when the phase is 0, and sketch it. 1 p.

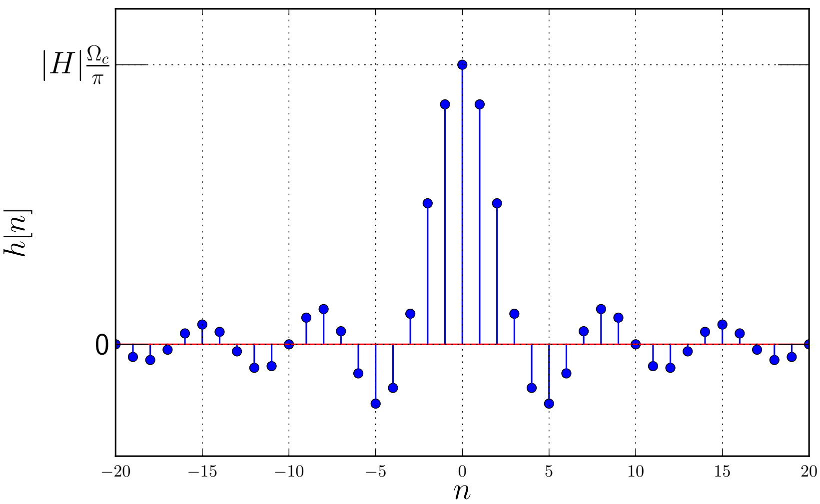
b) This filter can not be realised. Why? What must be done with the impulse response to make the filter realisable, and how does this affect the frequency response? 1 p.

c) Two common ways to design FIR filters is by means of the "frequency sampling method" and the "window method". Explain the concept of these two methods, and list pros and cons. 2 p.

Answer:

a) Finding $h_{\text{ideal}}[n]$ is simply a matter of realising that we need to use the inverse DTFT. Let us say that $|H_{\text{ideal}}(\Omega)| = A$ when $-\Omega_c < \Omega < \Omega_c$:

$$
\begin{aligned}
h_{\text{ideal}}[n] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{\text{ideal}}(\Omega)\, e^{j\Omega n} d\Omega \\
&= \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} A\, e^{j\Omega n} d\Omega \\
&= A\,\frac{1}{2\pi j n}\left[e^{j\Omega n}\right]_{-\Omega_c}^{\Omega_c} \\
&= A\,\frac{1}{2\pi j n}\left(e^{j\Omega_c n} - e^{-j\Omega_c n}\right) \\
&= A\,\frac{1}{2\pi j n}\, 2j\sin\Omega_c n \\
&= A\,\frac{1}{\pi n}\,\sin\Omega_c n \\
&= A\,\frac{\Omega_c}{\pi}\,\frac{\sin\Omega_c n}{\Omega_c n} \\
&= A\,\frac{\Omega_c}{\pi}\,\text{sinc}\,\Omega_c n
\end{aligned}
$$



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b) This impulse response (the sinc) has an infinite extent, and is therefore neither causal nor stable (it is not absolutely summable). What we can do, however, is to symmetrically truncate the impulse response at some point and see if the filter is still good enough. This operation corresponds to multiplying the impulserespons with a window function, or equvalently, convolving the frequency response with the frequency response of the window. This results in

- Ringing in the passband and stopband, and an overshoot/undershoot near the transition boundary (Gibbs effect).
- Transition band between passband and stopband, i.e., the transition edge becomes less steep.

c) **The frequency samping method**:

(1) Start with the frequency response (the DTFT) of the ideal filter.

(2) Sample this response at regular intervals and treat this as the DFT $H(f)$.

(3) Take the inverse DFT to find the filter coefficients/impulse response $h[n]$.

By taking the DFT of $h[n]$ one will see that the realized filter has a frequency response that passes through the sample points, but this method provides little means for controlling the response in between them. Introducing samples in the transition band helps.

The window method:

(1) Start with the frequency response (the DTFT) of the ideal filter.

(2) Take the inverse DTFT to get an impulse response $h[n]$ with infinite extent.

(3) Truncate the impulse response by multiplying it with a window.

When a rectangular windows is applied, the transition band is at its steepest, but ringing/ripple will occur in both pass- and stopband. A window that approaches 0 at the edges more "gently" will minimize ripple/ringing at the expense of a wider transition band. Finding the window that lead to the right compromise is what this method is all about.

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Problem 5 Filters

Given the system function:

$$H(z) = (1 - 1.25e^{j0.8\pi} z^{-1})(1 - 1.25e^{-j0.8\pi} z^{-1})$$

a) Find poles and zeros and plot them. 1 p.

b) Find the minimum-phase filter $H_{\mathrm{MF}}(z)$ that has the same frequency response:

$$H_{\mathrm{MF}}(\Omega) = H(\Omega)$$

1 p.

c) Find the allpass filter $H_{\mathrm{AP}}(z)$ that transforms between them, i.e.

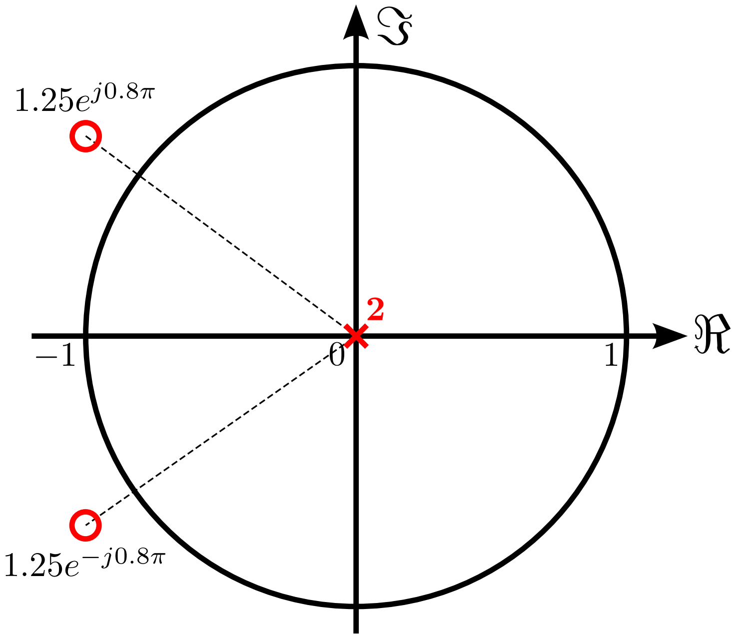
$$H_{\mathrm{MF}}(\Omega) = H_{\mathrm{AP}}(z)\, H(\Omega),$$

and determine whether this filter is stable. 1 p.

Answer:

a) Recognizing the poles and zeros is easy once we rewrite the term to get rid of the negative exponentials of z:

$$\begin{aligned} H(z) &= \left(1 - 1.25e^{j0.8\pi} z^{-1}\right)\left(1 - 1.25e^{-j0.8\pi} z^{-1}\right) \\ &= \frac{1}{z^2}\left(z - 1.25e^{j0.8\pi}\right)\left(z - 1.25e^{-j0.8\pi}\right) \end{aligned}$$



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b) The filter described in (a) is a maximum phase filter, since it is stable (FIR) and the zeros are outside the unit circle. By replacing the factors $(z-\alpha)$ with either $(\frac{1}{z}-\alpha)$ or $(1-\alpha z)$ we will get a minimum phase filter instead:

$$\begin{aligned} H_{\mathrm{MF}}(z) &= \frac{1}{z^2}\left(1-\tfrac{5}{4}e^{j0.8\pi}z\right)\left(1-\tfrac{5}{4}e^{-j0.8\pi}z\right) \\ &= \frac{1}{z^2}\frac{5}{4}e^{j0.8\pi}\frac{5}{4}e^{-j0.8\pi}\left(\tfrac{4}{5}e^{-j0.8\pi}-z\right)\left(\tfrac{4}{5}e^{j0.8\pi}-z\right) \\ &= \frac{5^2}{4^2}\frac{1}{z^2}\left(z-\tfrac{4}{5}e^{j0.8\pi}\right)\left(z-\tfrac{4}{5}e^{-j0.8\pi}\right) \end{aligned}$$

The text said to find the minimum phase filter with the same frequency response, $H_{\mathbf{MF}}(\Omega) = H(\Omega)$. It should have said <u>magnitude response</u>, $|H_{\mathbf{MF}}(\Omega)| = |H(\Omega)|$.

c) There's no magic here, really:

$$\begin{aligned} H_{\mathrm{AP}}(z) = \frac{H_{\mathrm{MF}}(z)}{H(z)} &= \frac{\frac{5}{4}^2\frac{1}{z^2}\left(z-\frac{4}{5}e^{j0.8\pi}\right)\left(z-\frac{4}{5}e^{-j0.8\pi}\right)}{\frac{1}{z^2}(z-\frac{5}{4}e^{j0.8\pi})(z-\frac{5}{4}e^{-j0.8\pi})} \\ = \frac{H_{\mathrm{MF}}(z)}{H(z)} &= \frac{5}{4}^2\frac{\left(z-\frac{4}{5}e^{j0.8\pi}\right)\left(z-\frac{4}{5}e^{-j0.8\pi}\right)}{(z-\frac{5}{4}e^{j0.8\pi})(z-\frac{5}{4}e^{-j0.8\pi})} \end{aligned}$$

If $1/H(z)$ is causal, the system will be unstable due to having poles on the outside of the unit circle.

(Continued on page 13.)

Formulas

Basic relations:

$$\begin{aligned}
\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\
\cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\
\sin 2\alpha &= 2\sin\alpha\cos\alpha \\
\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\
\sin\alpha + \sin\beta &= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\sin\alpha - \sin\beta &= 2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\
\cos\alpha + \cos\beta &= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\cos\alpha - \cos\beta &= -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\
\cos^2\alpha + \sin^2\alpha &= 1 \\
\cos\alpha &= \frac{1}{2}(e^{j\alpha} + e^{-j\alpha}) \\
\sin\alpha &= \frac{1}{2j}(e^{j\alpha} - e^{-j\alpha}) \\
\sum_{n=0}^{N-1} a^n &= \begin{cases} N & \text{for } a = 1 \\ \frac{1-a^N}{1-a} & \text{otherwise} \end{cases} \\
ax^2 + bx + c = 0 &\Leftrightarrow x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

Discrete convolution:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] = h[n] * x[n]$$

Discrete-time Fourier transformation (DTFT):

$$\begin{aligned}
\text{Analysis:} \qquad X(\Omega) &= = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n} \\
\text{Synthesis:} \qquad x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n} d\Omega
\end{aligned}$$

Discrete Fourier transformation (DFT):

$$\begin{aligned}
\text{Analysis:} \qquad X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}, \quad 0 \le k \le N-1 \\
\text{Synthesis:} \qquad x[n] &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N}, \quad 0 \le k \le N-1
\end{aligned}$$

z-transformation:

$$\text{Analysis:} \qquad X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$