UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in INF3470/4470 — Digital signal processing

Day of examination: December 12th, 2016

Examination hours: 14:30 – 18.30

This problem set consists of 16 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note 1: All numbers and figure axes should have units.
Note 2: Read through the whole exercise set before you start!

Svar:

Forslag til fasit, versjon-01:

Problem 1 Signals and systems. Various topics

a)
- Is the discrete sine function $\cos(\omega_0 n + \phi)$ always periodic? Justify your answer.
- Is $\cos(\pi^2 n + \pi/4)$ periodic? Justify your answer.

1.0 p.

b) Consider the system $y[n] = \log(|x[n]|)$. Is the system:

- Linear?
- Time invariant?
- Causal?
- Stable?

All answers must be justified! 1.0 p.

c) Consider the impulse response $h[n] = \{4, \underline{3}, 3, -2, 0\}$

- Is the filter causal?
- Express $h[n]$ using unit step functions
- Find the energy of $h[n]$
- Find $h[-n+1]$

(Continued on page 2.)

1.0 p.

d) Let us have a look at different types of responses:

- Explain the terms transient and stationary responses.
- What requirements must be met in order to apply frequency analysis (find $H(e^{j\omega})$) to a LTI system?
- Is it possible to use frequency analysis to find the transient response? Justify your answer.

1.0 p.

e) Signal distortion

- Explain what is meant by signal distortion.
- What does it mean that a system has a constant gain?
- What does it mean that a system has linear phase?
- Why is linear phase important to avoid signal distortion?

1.0 p.

Svar:

a)
- No, a discrete sinusoidal function $\cos(\omega_0 n + \phi) = \cos(2\pi f_0 n + \phi)$ is periodic if and only if the digital frequency f_0 may be written on the form $f_0 = k/N$, where both k and N are integers. Its period then equals N.
- The function $\cos(\pi^2 n + \pi/4)$ has a digital frequency $f_0 = \pi^2/(2\pi) = \pi/2$. This cannot be expressed as a ratio of integers, meaning that the function is not periodic.

b) We were to investigate the properties of the system $y[n] = \log(|x[n]|)$.

- Linearity: We have to check the superposition principle. That means we should require

$$\mathcal{H}\left\{a_1 x_1[n] + a_2 x_2[n]\right\} \stackrel{!}{=} a_1 y_1[n] + a_2 y_2[n]$$

The left side of the equation becomes:

$$\log(|a_1 x_1[n] + a_2 x_2[n]|)$$

The right side of the equation becomes: $a_1\log(|x_1[n]|) + a_2\log(|x_2[n]|)$ If any of you managed to get these sides equal, we will get deeply depressed. The system is not linear.

(Continued on page 3.)

- Time Invariant: A time invariant system implies that

$$y[n] = \mathcal{H}\{x[n]\} \Rightarrow y[n-n_0] = \mathcal{H}\{x[n-n_0]\}$$

We first apply a signal $x_1[n]$ to the system and get:

$$y_1[n] = \log(|x_1[n]|)$$

Then we define a shifted input signal $x_2[n] = x_1[n-n_0]$ and apply it:

$$y_2[n] = \log(|x_2[n]|) = \log(|x_1[n-n_0]|)$$

Delaying the output of $y_1[n]$ yields:

$$y_1[n-n_0] = \log(|x_1[n-n_0]|) = y_2[n]$$

A shift of n_0 on the input is equivalent to a shift n_0 on the output. The system is time invariant.

- Causality: The system is based only on current input, meaning it is causal.

- Stability: The criterion for stability is that a bounded input yields a bounded output:

$$|x[n]| \leq M_x < \infty \Rightarrow |y[n]| \leq M_y < \infty$$

Using basic knowledge of the logarithmic function, it should be evident that the bounded input $x[n] = 0$ yields $-\infty$ on the output. The system is unstable.

c) We are investigating the impulse response: $h[n] = \{4, \underline{3}, 3, -2, 0\}$

- No, $h[n] = 4$ for $n = -1$. It is based on future inputs.
- We have a step of 4 in $n = -1$, a step -1 in zero, a step -5 in $n = 2$ and a positive last step 2 in $n = 3$. This yields $h[n] = 4u[n+1] - u[n] - 5u[n-2] + 2u[n-3]$.
- The energy of $h[n]$ is found as $E = 4^2 + 3^2 + 3^2 + (-2)^2 = 38$.
- We do shifting and folding: $h[-n+1] = \{0, -2, \underline{3}, 3, 4\}$

d)

- The transient response is the part of the total response that diminishes with time for a stable system excited with a periodic or constant input signal, for instance a unit step. The stationary response is the system response after the all the transients have died out. Examples of transient behavior can be overshoots and ringing in filters, or filter behavior of not yet fully initialized filters.
- In order to perform frequency analysis on a LTI system, the system has to be stable. Frequency analysis assumes all transients have died out, as it is founded on analysing systems having reached their steady state.

(Continued on page 4.)

- No. Frequency analysis will only give you the steady state response.

e) Signal distortion

- By signal distortion, we mean that the *shape* of the input signal is distorted. This can either be because different frequencies are attenuated differently (amplitude distortion), or because different frequencies are delayed differently (phase distortion).
- A system having constant gain has no amplitude distortion. All frequency components are multiplied by the same amount, $|H(e^{j\omega})| = K$.
- A system having linear phase has $\angle H(e^{j\omega}) = -\omega n_d$.
- Linear phase causes all frequency components to be equally delayed in the filer. This means we avoid signal distortion. This is evident by investigating the general LTI case:

$$\begin{aligned} y[n] &= A_x |H(e^{j\omega})| \cos(\omega n + \phi_x + \angle H(e^{j\omega})) \\ y[n] &= A_x |H(e^{j\omega})| \cos(\omega(n + \phi_x/\omega + \angle H(e^{j\omega})/\omega) \end{aligned}$$

The term $\angle H(e^{j\omega})/\omega$ acts as a time delay. If $\angle H(e^{j\omega})$ is linear the delay becomes constant for all frequencies.

Svar:

Forslag til fasit, versjon-01:

Problem 2 Frequency analysis of LTI systems

a) Assume a difference equation $y[n] - ay[n-1] = x[n] - bx[n-1]$, where a and b are real constants.

- Find the transfer function, $H(z)$, of the system.
- When is the system causal, stable and minimum phase?

1.5 p.

b) Given that the filter in a) is causal and stable.

- Elaborate on the magnitude response of the filter (low pass, high pass, all pass) for different choices of the constant b, when you can assume $a = 0.8$. You may use pole zero plot(s) to explain your answer.

1.5 p.

c) Given that the filter in a) is causal and stable.

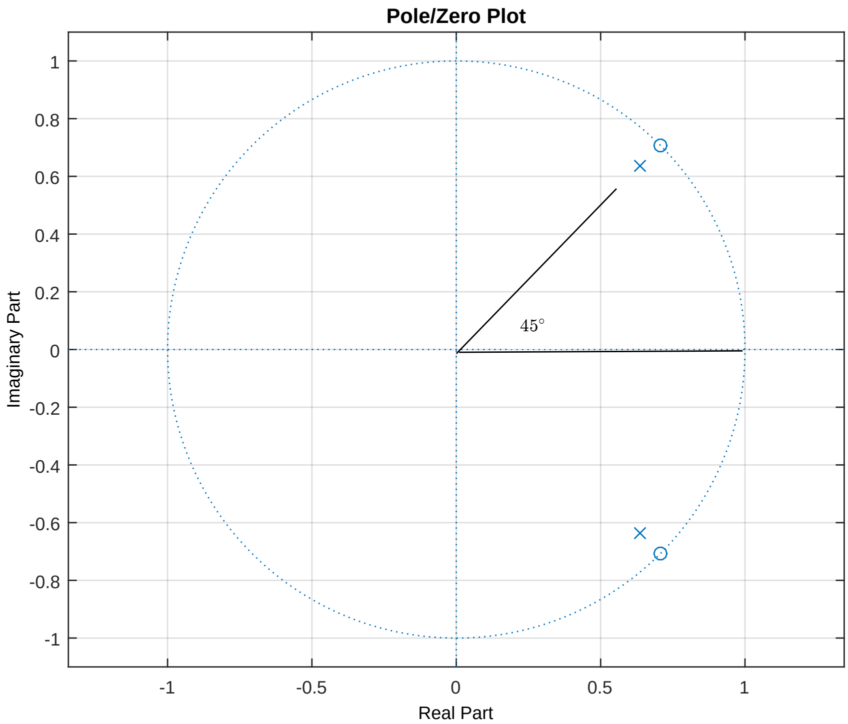
(Continued on page 5.)

- Show that the impulse response is given by

$$h[n] = \begin{cases} 0, n < 0 \\ 1, n = 0 \\ (a-b)a^{n-1}, n > 0. \end{cases}$$

Tip: The system may be written as a sum of two very simple systems, where one system is merely a scaled and delayed version of the other. 1.0 p.

d) Assume that you have a system with the pole zero plot shown below.



A continuous signal $x(t)$ is sampled at the sampling frequency $F_s = 1$ kHz, giving the discrete signal $x[n]$, which is applied to the system.

- What type of filter is this?
- Which physical frequency component (measured in Hz) of the input signal will be filtered most strongly by the filter?

1.0 p.

Svar:

a)
- $y[n] - ay[n-1] = x[n] - bx[n-1]$ gives us the following transfer function:

$$\begin{aligned} Y(z) - az^{-1}Y(z) &= X(z) - bz^{-1}X(z) \\ H(z) = \frac{Y(z)}{X(z)} &= \underline{\underline{\frac{1-bz^{-1}}{1-az^{-1}}}} \end{aligned}$$

(Continued on page 6.)

- The system is stable and causal is all poles of the transfer function is within the unit circle, and has a ROC pointing outwards. This implies $|a| < 1$.
  The system is minimum phase if all poles AND zeros are within the unit circle, which implies $|a| < 1$ and $|b| < 1$. Note that both a and b are real constants (as informed in the exercise).

b) The contribution to the magnitude response from the poles and zeros is dependent on the distance from the pole/zero to the point on the unit circle representing the frequency you are investigating. A long vector from a zero to the point on the unit circle, yields a strong response for that frequency. A short vector gives a weak response. A pole gives the opposite effect (as the contribution is in the numerator). Here we have a pole on the positive real axis at $z = 0.8$ and a zero at b, also on the real axis. As long as the zero is farther away from the unit circle than the pole, the response will be dominated by the pole. The real axis corresponds to the 0 frequency (DC).

- If the zero is to the left of the pole ($b < 0.8$) or to the right of a fictive pole mirrored about the unit circle ($1 + (1 - 0.8) = 1.2$), ($b > 1.2$), the pole will dominate and we will have a strong response for low frequencies. We have a low pass natured filter for $b < 0.8$ or $b > 1.2$.

- If the zero is between the pole and the fictive mirrored pole at z=1.2, (that is $0.8 < b < 1.2$), the contribution from the zero will be stronger than from the pole, pulling the low frequency response downwards. We have a high pass filter for $0.8 < b < 1.2$.

- If the zero is the complex reciprocal of the pole, ($b = 1/0.8 = 1.25$), we should recognize the case of an allpass filter. This means the magnitude response is constant for all frequencies. This happens when the *ratio* between the distance from the pole to a point on the unit circle and the distance from the zero to the same point on the unit circle, is constant for all angles/frequencies. We have an allpass filter for $b = 1/0.8$.

c) We are using that $H(z)$ is causal and stable.

$$H(z) = \frac{1 - bz^{-1}}{1 - az^{-1}} = \frac{1}{1 - az^{-1}} - \frac{bz^{-1}}{1 - az^{-1}} = H_1(z) - H_2(z) \qquad (1)$$

This should be recognizable as the sum of two simple first order systems, where the second system,$H_2(z)$ is merely a scaled and 1 sample delayed version of $H_1(z)$.

(Continued on page 7.)

$$h_1[n] = \begin{cases} 0, n < 0 \\ a^n, n \geq 0. \end{cases} \tag{2}$$

$$h_2[n] = \begin{cases} 0, n \leq 0 \\ bh_1[n-1] = ba^{(n-1)}, n > 0. \end{cases} \tag{3}$$

which yields

$$h[n] = \begin{cases} 0, n \leq 0 \\ a^n, n = 0 \\ (a-b)a^{n-1}, n > 0. \end{cases} \tag{4}$$

d)
- The filter is an IIR type Notch filter. The zeros on the unit circle filters out the corresponding frequency, and the sharpness of the notch filter is improved by the poles close to the unit circle (at the same angle).
- The poles are at an angle ± 45 degrees, or $\omega = \pi/4$ radians. In digital frequency this corresponds to $f = \omega/(2\pi) = 1/8$. In physical frequency this is $F = f * F_s = 1/8 * 1000 = 125$ [Hz]. The filter will filter out the frequency components corresponding to 125 [Hz].

Svar:

Forslag til fasit, versjon-01:

Problem 3 Sampling

Consider the signal

$$s(t) = 5\cos(20\pi t) + 10\cos(12\pi t).$$

It is sampled at frequency $F_s = 15$ Hz.

a) Will there be aliasing? Justify your answer. 0.5 p.

b) If the signal is sampled at F_s using a perfect analogue to digital converter and reconstructed using the ideal band-limited interpolator, give an expression for the reconstructed signal. 0.5 p.

c) Give an expression for the sampled signal $s[n]$. 0.5 p.

d) Consider the sampled signal $s[n]$ for $0 \leq n \leq 29$. Compute the signal's DFT $S[k]$ and plot it. Remember axis labelling. 1 p.

e) Using the result from d) compute the inverse DFT of $S[k]$. Show that the obtained result is the sampled version of the signal obtained in b). 0.5 p.

(Continued on page 8.)

Svar:

a) The signal can be writen as

$$s(t) = 5\cos(2\pi 10t) + 10\cos(2\pi 6t). \tag{5}$$

It is made of two signals of frequency 10 Hz and 6 Hz. All frequencies above $F_s/2 = 7.5$ Hz will be aliased according to the sampling theorem. The signal will therefore be aliased.

b) The part of the signal at 10 Hz will be aliased and will be reconstruced with an apparent frequency $F_a = 10 - F_s = -5$ Hz. The reconstructed signal is therefore

$$s_r(t) = 5\cos(10\pi t) + 10\cos(12\pi t). \tag{6}$$

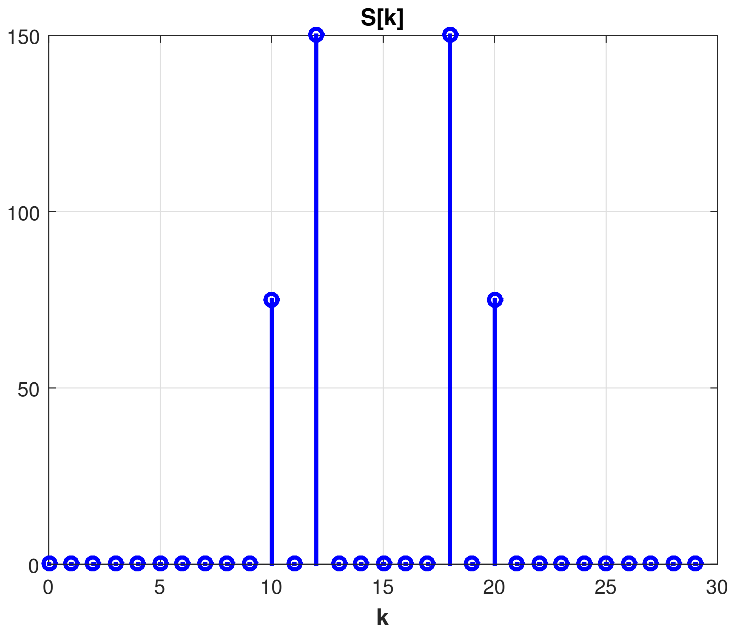
c) The sampled signal is the signal obtained at $t = nT$ where $T = 1/F_s = 1/15$.

$$s[n] = 5\cos\left(2\pi\frac{10n}{15}\right) + 10\cos\left(2\pi\frac{6n}{15}\right) \tag{7}$$

d) We compute the 30-points DFT of $s[n]$

$$\begin{aligned}
S[k] &= \sum_{n=0}^{29} s[n]e^{-j2\pi kn/30} \\
&= \sum_{n=0}^{29}\left(5\cos\left(2\pi\frac{10n}{15}\right) + 10\cos\left(2\pi\frac{6n}{15}\right)\right)e^{-j2\pi kn/30} \\
&= \frac{1}{2}\sum_{n=0}^{29}\left(5e^{j2\pi 10n/15} + 5e^{-j2\pi 10n/15} + 10e^{j2\pi 6n/15} + 10e^{-j2\pi 6n/15}\right)e^{-j2\pi kn/30} \\
&= \frac{5}{2}\sum_{n=0}^{29}\left(e^{j2\pi[(20-k)n/30]} + e^{-j2\pi[(20+k)n/30]}\right) + \frac{10}{2}\sum_{n=0}^{29}\left(e^{j2\pi[(12-k)n/30]} + e^{-j2\pi[(12+k)n/30]}\right) \\
S[k] &= 30\frac{5}{2}(\delta[20-k] + \delta[10-k]) + 30\frac{10}{2}(\delta[12-k] + \delta[18-k]) \quad 0 \le k \le 29
\end{aligned}$$

(Continued on page 9.)



e) Taking the inverse DFT of the previous signal, we get

$$
\begin{aligned}
s_r[n] &= \frac{1}{30}\sum_{k=0}^{29} S[k]e^{j2\pi kn/30} \\
&= \frac{5}{2}\left(e^{j2\pi 20n/30} + e^{j2\pi 10n/30}\right) + \frac{10}{2}\left(e^{j2\pi 12n/30} + e^{j2\pi 18n/30}\right) \\
&= \frac{5}{2}\left(e^{-j2\pi 10n/30} + e^{j2\pi 10n/30}\right) + \frac{10}{2}\left(e^{j2\pi 12n/30} + e^{-j2\pi 12n/30}\right) \\
&= 5\cos\left(\frac{2\pi 10n}{30}\right) + 10\cos(\left(\frac{2\pi 12n}{30}\right) \\
&= 5\cos\left(\frac{2\pi 5n}{15}\right) + 10\cos(\left(\frac{2\pi 6n}{15}\right)
\end{aligned}
$$

which is the signal $5\cos(10\pi t) + 10\cos(12\pi t)$ sampled at frequency $F_s = 1/T = 15$ Hz.

Svar:

Forslag til fasit, versjon-01:

Problem 4 Filters

Consider the design of a filter to remove all frequencies greater than 150 Hz from a signal $s(t)$. Frequencies below 150 Hz shall be unchanged.

a) The signal is sampled at $F_s = 600$ Hz. Draw the ideal (not realizable) filter that would fulfill the above requirements. Use the normalized frequency ω and remember axis labeling. 0.5 p.

(Continued on page 10.)

b) We want to design a practical FIR filter from the ideal filter using fixed windows. We allow ripple levels of $\delta_p = 0.01$ in the passband and $\delta_s = 0.005$ in the stopband. What fixed window(s) listed in the table under can we use for this filter? 0.5 p.

Window name	Side lobe level [dB]	Mainlobe width	Transition band width ($\Delta\omega$)	Ripple level ($\delta_p \approx \delta_s$)	A_p [dB]	A_s [dB]
Rectangular	-13	$4\pi/L$	$1.8\pi/L$	0.09	1.57	21
Bartlett	-25	$8\pi/L$	$6.1\pi/L$	0.05	0.87	26
Hann	-31	$8\pi/L$	$6.2\pi/L$	0.0063	0.11	44
Hamming	-41	$8\pi/L$	$6.6\pi/L$	0.0022	0.038	53
Blackman	-57	$12\pi/L$	$11\pi/L$	0.0002	0.0035	74

Table 1: Characteristics of fixed windows used for FIR filter design. L represents the length of the filter's impulse response.

c) The specified passband and stopband frequencies are $\omega_p = 0.5\pi$ and $\omega_s = 0.6\pi$. What is the minimum filter order that can be obtained? 0.5 p.

d) The stopband frequency ω_s is changed from 0.6π to 0.55π, what becomes the minimum filter order that can be obtained? What is the consequence on the ripple levels? 0.5 p.

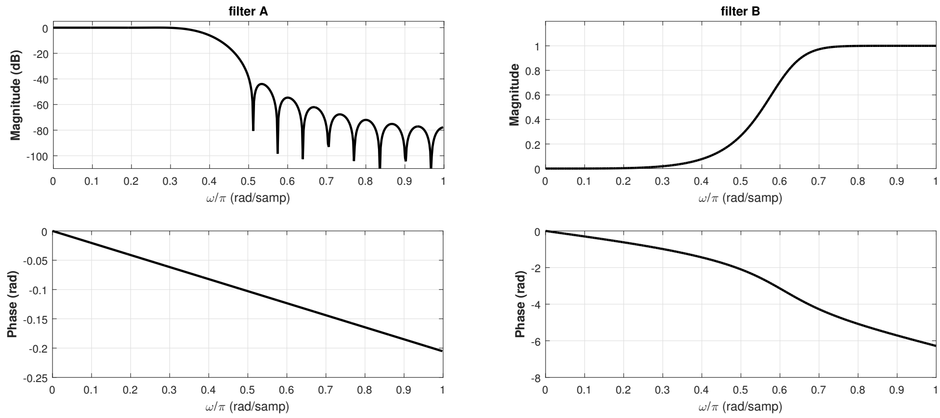
e) Consider a different FIR filter design not using the fixed windows above. A frequency response with equiripple both in the passband and stopband is requested. What kind of FIR filter should we choose? Would the order of this filter be smaller, equal, or larger than the order for the previous filter? 0.5 p.

f) We still require a frequency response with equiripple both in the passband and stopband but use IIR filters instead of FIR filters. What type of filter should we choose? How would its order compare to the order of the filter designed in e) (smaller, equal, or larger)? What is the trade-off one has to accept when switching from FIR to IIR filters? 0.5 p.

g) The figure below shows the magnitude and phase of the frequency response for two filters. Describe in as much details as possible what kind of filters they represent. Justify your answers.

- lowpass, basspand, highpass, or stopband
- FIR (symmetric) or IIR

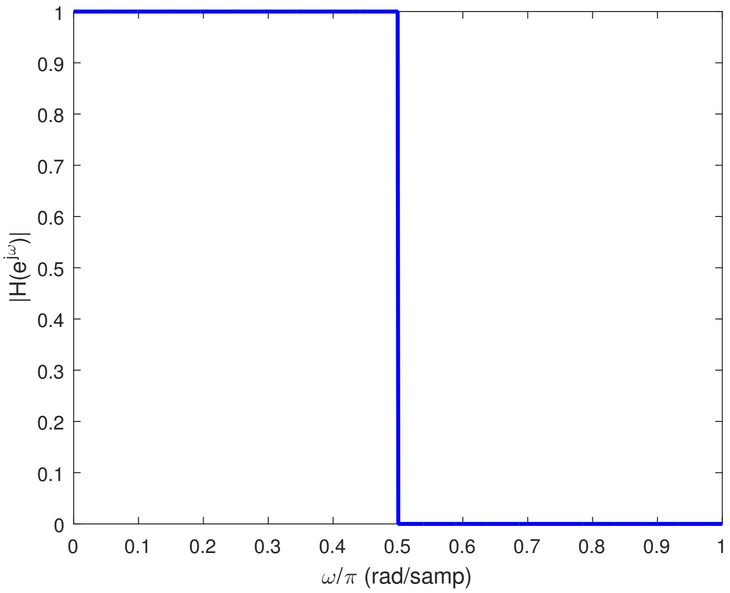
(Continued on page 11.)



1 p.

Svar:

a) The ideal filter is a "brick-wall" filter with a cutoff frequency $F_c = 150$ Hz corresponding to the normalized frequency $\omega_c = 2\pi F_c/F_s = 2\pi 150/600 = 0.5\pi$



b) The minimum ripple level is $\delta = \min(\delta_p, \delta_s) = 0.005$. According to the given table, the Hamming and Blackmann windows satisfy those specifications and could be used for this filter.

c) The transition band width is $\Delta\omega = \omega_p - \omega_s = 0.1\pi$. Again according to the given table, a Hamming and Blackmann window would give a filter length (L) equal to 66 and 110, respectively. The minimum filter order that can be obtained is therefore $M = L - 1 = 65$ and is obtained using a Hamming window. If we want to use a type I filter, the minimum filter order would be $M = 66$.

d) The transition bandwith is now $\Delta\omega = 0.05\pi$. The minimum filter length is therefore multiplied by two and $L = 132$, $M = 131$. Again if we want a type I filter, the minimum filter order would be $M = 132$.

(Continued on page 12.)

The ripple level is not affected by increasing the filter order. The ripple is defined by the window choice, changing the filter order only influences the transition band.

e) To get an equiripple frequency response, we need to use a Chebyshev type filter. The order of the filter will be smaller than the previous filter's order since Chebyshev type FIR filters have a lower order than fixed windows FIR filters. They are more efficient at exploiting the allowed ripple in the pass and stop bands.

f) An equiripple IIR filter is obtained using an elliptic filter design. Its order will be much smaller than the previous filter's order as IIR filters have much lower order than FIR filter for given specifications.

The trade-off when using an IIR filter instead of an FIR is that the phase cannot be linear. FIR filter can be designed to have linear phase whereas IIR cannot.

g) Filter A is a lowpass filter. It is an FIR filter with linear phase. It is not possible from the frequency response to tell what type of FIR filter it is (I, II, III, or IV).

Filter B is a highpass IIR filter. It can be seen from its nonlinear phase response. The filter is a Butterworth filter (maximally flat response).

Svar:

Forslag til fasit, versjon-01:

Problem 5 Digitizing and digital processing

Consider the following voltage signal:

$$s(t) = s_1(t) + s_2(t) + s_3(t)$$

where

$$\begin{aligned} s_1(t) &= 5\sin(1000\pi t + \pi/3) \\ s_2(t) &= 2\sin(900\pi t) \\ s_3(t) &= 3\cos(400\pi t). \end{aligned}$$

We wish to digitize it and study it in the frequency domain. The signal $s(t)$ has unit $[V]$.

a) What analogue stage/component is it common practice to place before an analogue to digital converter (ADC)? What is its purpose? 0.5 p.

b) Assuming the analogue range of the ADC is ± 10 V, what is the minimum number of bits we should use for the ADC in order to keep the quantization error $|e|$ to 0.1 V or lower. The quantization step is done using the rounding operation. 1 p.

(Continued on page 13.)

c) Determine the minimum sampling frequency F_s to properly sample $s(t)$ 0.5 p.

d) Assume a sampling frequency $F_s = 1$ kHz. We record $s(t)$ for 10 ms and obtain the digitized signal $s[n]$. By taking the DFT of $s[n]$ will you be able to identify all the frequency components in $s(t)$ from its frequency spectrum magnitude? Justify your answer. 1 p.

e) Before taking the DFT of $s[n]$ we now multiply it with a Hann window of equal length $w[n]$. Explain the consequences on the frequency spectrum of $s[n] \cdot w[n]$ compared to that of $s[n]$. 0.5 p.

f) We wish to reconstruct the analogue signal from the digitized signal. Since the ideal band-limited interpolator is not realizable, we use the sample and hold amplifier with impulse response

$$g_{SH}(t) = 1 \quad \text{if } 0 \le t \le T, \quad 0 \text{ otherwise}$$

where $T = 1/F_s$. What are the undesirable consequences of using this filter on the frequency spectrum of the reconstructed signal? How can we compensate for them? 1.0 p.

Svar:

a) It is common practice to place a lowpass analogue filter before digitizing a signal. This ensures that all frequencies larger that the Nyquist frequency are suppressed since they would create aliasing otherwise.

b) Since the ADC input analogue voltage range $V_{in} = 20$ V (± 10 V), the quantization step of a b-bit ADC is $\Delta = V_{in}/2^b$. Since quantization is using the rounding operation the quantization error is $-\Delta/2 \le e \le \Delta/2$, that is $|e| \le V_{in}/2^{b+1}$. For us, this means that we need $2^{b+1} \ge V_{in}/|e|$. The minimum value for b is 7 and is obtained when $V_{in} = 20$ V: $2^{b+1} = 2^8 = 256 > 20/0.1$. The ADC must therefore have at least 7 bits.

c) The frequencies of $s_1(t)$, $s_2(t)$,and $s_3(t)$ are 500 Hz, 450 Hz, and 200 Hz, respectively. According to the sampling theorem to avoid aliasing and to properly sample $s(t)$ the sampling frequency F_s must be equal or greater than twice the signal's highest frequency. In our case this means $F_s \ge 1$ kHz.

d) Recording 10 ms of $s(t)$ at $F_s = 1000$ Hz, we get $N = 0.01 * 1000 = 10$ samples for $s[n]$. A 10-point DFT will therefore give a frequency spacing of $F_s/N = 100$ Hz between each value. The frequencies of s_1 and s_2 being separated by only 50 Hz, we will not be able to identify them separately from the frequency spectrum magnitude.

(Continued on page 14.)

e) Multiplying $s[n]$ by a Hann window $w[n]$ can be seen as replacing a rectangular window with a Hann window. The two effects of windowing are smearing and leakage. Replacing the rectangular window with a Hann window will increase the smearing of single frequency components (larger mainlobe width) but decrease the leakage of energy to other frequencies (lower sidelobe level).

f) Using the sample and hold amplifier the two main drawbacks on the frequency spectrum of the reconstructed signal are that the replicas due to aliasing are not totally removed due to the sidelobes of its Fourier transform and that the gain is not constant over the frequency range $|F| \leq |F_s/2|$ (droop effect).

To compensate for this, we can use an analogue lowpass filter that compensate for the droop effect in the frequency bandwidth $|F| \leq |F_s/2|$ and sets all other frequencies to 0.

(Continued on page 15.)

Formula sheet

Basic relations:

$$\begin{aligned}
\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\
\cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\
\sin 2\alpha &= 2\sin\alpha\cos\alpha \\
\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\
\sin\alpha + \sin\beta &= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\sin\alpha - \sin\beta &= 2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\
\cos\alpha + \cos\beta &= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\cos\alpha - \cos\beta &= -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\
\cos^2\alpha + \sin^2\alpha &= 1 \\
\cos\alpha &= \frac{1}{2}(e^{j\alpha} + e^{-j\alpha}) \\
\sin\alpha &= \frac{1}{2j}(e^{j\alpha} - e^{-j\alpha}) \\
\sum_{n=0}^{N-1} a^n &= \begin{cases} N & \text{for } a = 1 \\ \frac{1-a^N}{1-a} & \text{otherwise} \end{cases} \\
ax^2 + bx + c = 0 &\Leftrightarrow x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

Linear convolution:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] = h[n] * x[n]$$

Circular convolution:

$$y[n] = x[n] \circledN h[n] = \sum_{k=0}^{N-1} x[k]h[< n-k >_N] = \sum_{k=0}^{N-1} x[< n-k >_N]h[k] = h[n] \circledN x[n]$$

Discrete Time Fourier Transform (DTFT):

$$\begin{aligned}
\text{Analysis:} \qquad X(\Omega) &= = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n} \\
\text{Synthesis:} \qquad x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n} d\Omega
\end{aligned}$$

Discrete Fourier Transform (DFT):

$$\begin{aligned}
\text{Analysis:} \qquad X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}, \quad 0 \le k \le N-1 \\
\text{Synthesis:} \qquad x[n] &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N}, \quad 0 \le k \le N-1
\end{aligned}$$

(Continued on page 16.)

z-transform:

$$\text{Analysis:} \qquad X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$