UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Eksamen i	INF3470/4470 — Digital signalbehandling
Eksamensdag:	10. desember 2018
Tid for eksamen:	14:30 – 18.30
Oppgavesettet er på 19 sider.	
Vedlegg:	Ingen
Tillatte hjelpemidler:	Ingen

Kontroller at oppgavesettet er komplett før du begynner å besvare spørsmålene.

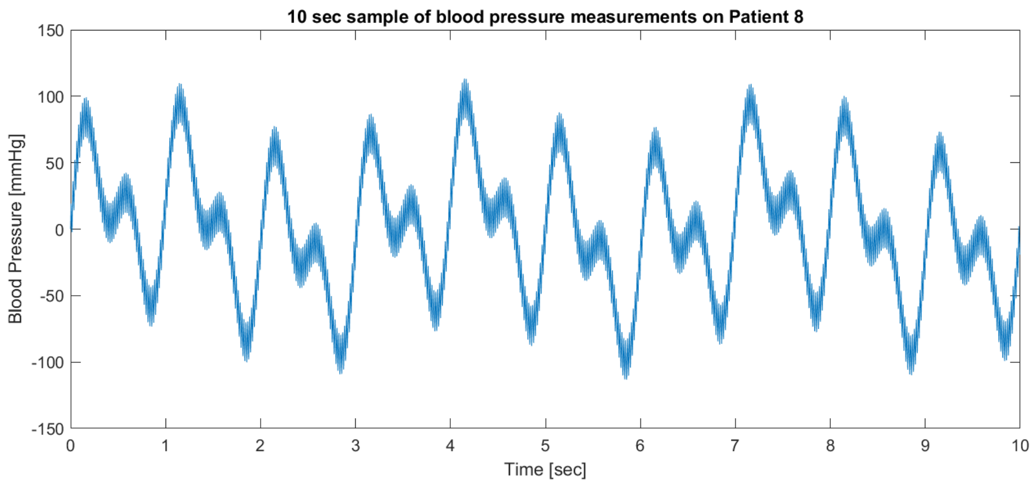
Merknad 1: Alle størrelser og figurakser skal benevnes.
Merknad 2: Les gjennom hele oppgavesettet før du begynner!

Svar:

Forslag til fasit, versjon-01:

Oppgave 1 Blodtrykksmålinger (13 p.)

Marit er mastergradstudent og har fått et datasett av veilederen sin. Hun skal analysere blodtrykksmålinger utført på pasienter. Figur 1 viser signalet

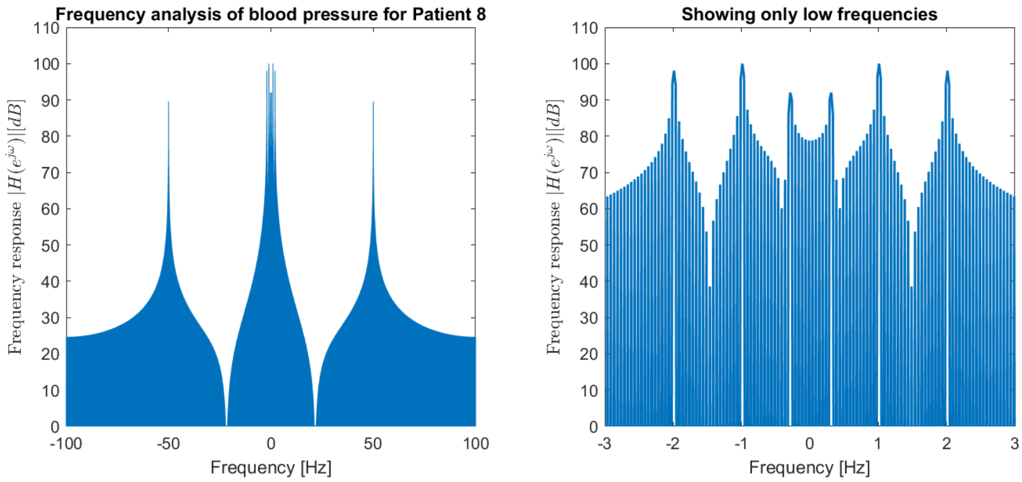


Figur 1: Utdrag fra blodtrykksmålinger til en av pasientene

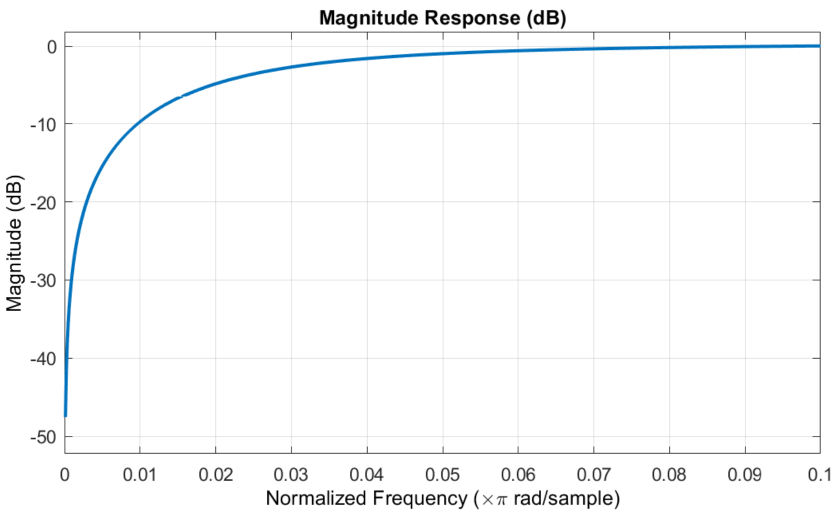
fra en av pasientene. Samplingsfrekvensen er 200 Hz. Målingene er støyete og veilederen forklarer at de sikkert inneholder bidrag fra både pasienten sin pust og støy fra selve opptaksutstyret. Pustefrekvens heter respirasjonsfrekvens

(Fortsettes på side 2.)

på det medisinske fagspråket, og betyr antall innåndinger per minutt. Marit tar Fourieranalyse av signalet. Figur 2 viser resultatet, hvor vi kan se fire dominante frekvenser.



Figur 2: Frekvensanalyse av blodtrykksmålingene i Figur 1. De fire dominante frekvensene er 0.3, 1, 2 og 50 Hz.



Figur 3: Magnituderespons til Marits filter i b). Figuren er plottet i MATLAB. Merk at MATLAB normaliserer frekvens med Nyquistfrekvensen istedenfor samplingsfrekvensen.

a) For en voksen person er pustefrekvensen normalt rundt 12-16 innåndinger per minutt. Pustefrekvensen til denne pasienten er 0.3 Hz. Hvor mange innåndinger per minutt tilsvarer det? Vis utregning. 1 p.

b) For å kunne fjerne pustens bidrag ser Marit først på et enkelt, lineært filter som har differanseligningen $y[n] + ay[n-1] = x[n] + bx[n-1]$.

(Fortsettes på side 3.)

- Finn systemets transferfunksjon H(z). 1 p.
- For hvilke verdier av a og b er filteret stabilt? Når er filteret kausalt? Når har filteret minimum fase? Begrunn svarene. 3 p.
- Marit velger a og b verdier til filteret sitt. Hun plotter magnituderesponsen til filteret i MATLAB for positive normaliserte frekvenser fra 0 til 0.1 og får Figur 3. Merk at MATLAB normaliserer med Nyquistfrekvensen, $F_s/2$, istedenfor samplingsfreksvensen. Normalisert Nyquistfrekvens tilsvarer i dette tilfellet altså 1. Hvordan vil dette filteret påvirke de dominante frekvensene vi ser i Figur 2 og det resulterende filtrerte signalet? Begrunn svaret. 2 p.

c) For at signalet skal gi mening for leger å analysere bør bidraget fra det ufysiologiske støyet på 50 Hz fjernes. Du får i oppgave å hjelpe Marit med å designe et notch filter som fjerner dette 50 Hz-støyet. Filteret skal ha to poler og to nullpunkter. Filteret skal være reelt.

- Oppgi plassering for poler og nullpunkter. Inkluder pol-nullpunktsplott. Begrunn valg av vinkel og radius for poler og nullpunkter. Ha tydelige aksebenevninger. 2 p.
- Oppgi systemets transferfunksjon H(z). 1 p.
- Skisser magnituderesponsen til filteret ditt. Ha tydelige aksebenevninger. 1 p.
- Istedenfor å lage dette notch filteret, hva er det enkleste FIR-filteret du kunne ha laget for å filtrere 50 Hz-støyet? Filteret skal ha reelle koeffisienter. Finn H(z) og tegn pol-nullpunktsplottet. Begrunn svaret. 2 p.

Svar:

a) A frequency of 0.3 Hz is the same as 0.3 $s^{-1} = \frac{0.3}{s}\frac{60s}{min} = \frac{18}{min}$. This patient therefore has 18 inhalations per minute.

b)
- The system transfer function H(z) is found by first taking the z-transform of the difference equation:

$$Y(z) + az^{-1}Y(z) = X(z) + bz^{-1}X(z)$$
$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \underline{\frac{(1+bz^{-1})}{(1+az^{-1})} = \frac{z+b}{z+a}}$$

- The system is stable if all the poles of the z-transform are within the unit circle, i.e. $\underline{\underline{|a| < 1}}$. It is causal if the ROC extends outwards, i.e. $\underline{\underline{|z| > |a|}}$. The system is therefore both stable and causal if $|a| < 1$ and $|z| > |a|$. The system has minimum phase if all poles AND zeros are within the unit circle, which means $\underline{\underline{|a| < 1 \text{ og } |b| < 1}}$.

(Fortsettes på side 4.)

- The respiration frequency of 0.3 Hz lies very close to the two dominant frequencies at 1 and 2 Hz. When the sampling frequency is as high as 200 Hz, the normalized frequency for all three will be very close to zero. The normalized frquencies are $\frac{F_0}{F_s} = \frac{0.3}{200} = 0.0015$, $\frac{1}{200} = 0.005$ and $\frac{2}{200} = 0.01$. The angular frequencies are $2\pi\frac{F_0}{F_s} = 2\pi\frac{0.3}{200} = 0.003\pi$, $2\pi\frac{1}{200} = 0.01\pi$ and $2\pi\frac{2}{200} = 0.02\pi$ respectively. It will therefore be difficult to make a filter that manages to only suppress the 0.3 Hz frequency.

  Figure 3 shows the magnitude response of the filter with MATLAB-normalized frequencies on the x-axis. As stated in the text, normalized frequency in MATLAB is the frequency normalized by the Nyquist frequency instead of the sampling frequency, i.e. $f_{matlab} = F_0/(F_s/2)$. Angular frequency in MATLAB is therefore defined as $\pi f_{matlab} = \pi\frac{F_0}{F_s/2}$ instead of $2\pi f = 2\pi\frac{F_0}{F_s}$. Our three frequencies in MATLAB-normalized frequency are $\frac{F_0}{F_s/2} = \frac{0.3}{200/2} = 0.003$, 0.01 and 0.02. From Figure 3 we see that the 1 Hz and 2 Hz frequencies will be suppressed by -10 dB and -5 dB respectively. Ideally these should not be suppressed at all.

  Grading: Reasoning should mention either very similar normalized frequencies or angles very close to zero in a filter's pole-zero plot for these three dominant frequencies. If this is not given, the importance of the sampling frequency has not been discussed and points will be deducted. For full score, the answer must include comment on the magnitude suppression for the 1 and 2 Hz frequencies given in the figure.

c)

- The two zeros should be placed at the normalized frequency corresponding to the 50 Hz noise, i.e. $f = \frac{50}{200} = 0.25$. In angular frequency, this corresponds to angle $\phi = 2\pi\frac{F_0}{F_s} = 2\pi\frac{50}{200} = \frac{\pi}{2} = 90^\circ$ in the pole-zero plot. This will ensure complete suppression of the 50 Hz noise since the numerator in $H(z)$ then will equal zero. The zeros are therefore located at

$$z_{1,2} = \pm j = e^{\pm j\frac{\pi}{2}}$$

  In the notch filter, the two poles are located at the same angle as the two zeros in the pole-zero plot. The poles are located here to minimize the suppression from the zero for the surrounding frequencies. In this example, the 50 Hz frequency is very far away from the three other dominant frequencies. A low value for the pole radii can therefore be chosen. Choosing a radius close to 1 will create a steeper magnitude response at the 50 Hz frequency, i.e. decreased suppression of surrounding frequencies. In this example, such a steep magnitude drop might not fully suppress all the contribution from the 50 Hz noise that we see in Figure 2. The radius should therefore ideally be chosen to be less than 0.9. If the

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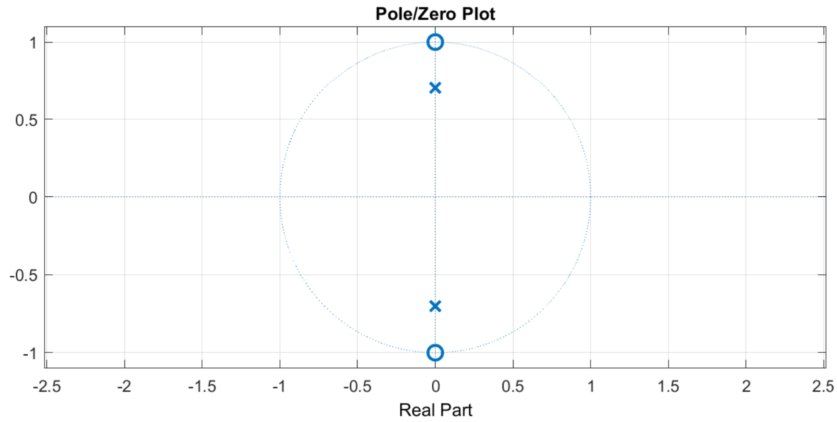
radius is for example chosen to be 0.7, the poles are placed at:

$$p_{1,2} = \pm 0.7j = 0.7e^{\pm j\frac{\pi}{2}}$$

The pole-zero plot of the notch filter with pole radii at 0.7 is given in Figure 4.



Figur 4: Pole-zero plot for notch filter

Grading: 1p. for correct zeros with reasoning. 1p. for correct poles and reasoning. Reasoning for the pole radii should also mention distance from the other frequencies with respect to radius and the resulting amount of suppression.

- For a notch filter with pole radius at 0.7, the transfer function is:

$$H(z) = \frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)} = \frac{(z-j)(z+j)}{(z-0.7j)(z+0.7j)} = \underline{\frac{z^2+1}{z^2+0.7^2}}$$

$$H(z) = \frac{(1-e^{j\frac{\pi}{2}}z^{-1})(1-e^{-j\frac{\pi}{2}}z^{-1})}{(1-0.7e^{j\frac{\pi}{2}}z^{-1})(1-0.7e^{-j\frac{\pi}{2}}z^{-1})}$$
$$= \frac{(1-2\cos(\frac{\pi}{2})z^{-1}+z^{-2})}{(1-1.4\cos(\frac{\pi}{2})z^{-1}+0.7^2z^{-2})} = \underline{\frac{1+z^{-2}}{1+0.7^2z^{-2}}}$$

Figure 5 shows the filtered signal's frequency response after filtering with a notch filter with pole radius of 0.7 and 0.95, for comparison.

- The magnitude response of the notch filter is shown in Figure 6.

Grading: The magnitude response must sketched as a filter with a very narrow stopband at the 50 Hz noise and have almost no suppression for the other frequencies. The x-axis must be clearly given and should be either positive and negative normalized frequencies or angular frequencies in the range $-\pi$ to π or 0 to 2π.

- If the pole radii is chosen to be zero in a notch filter, the filter becomes a simple Finite Impulse Response (FIR) filter with two

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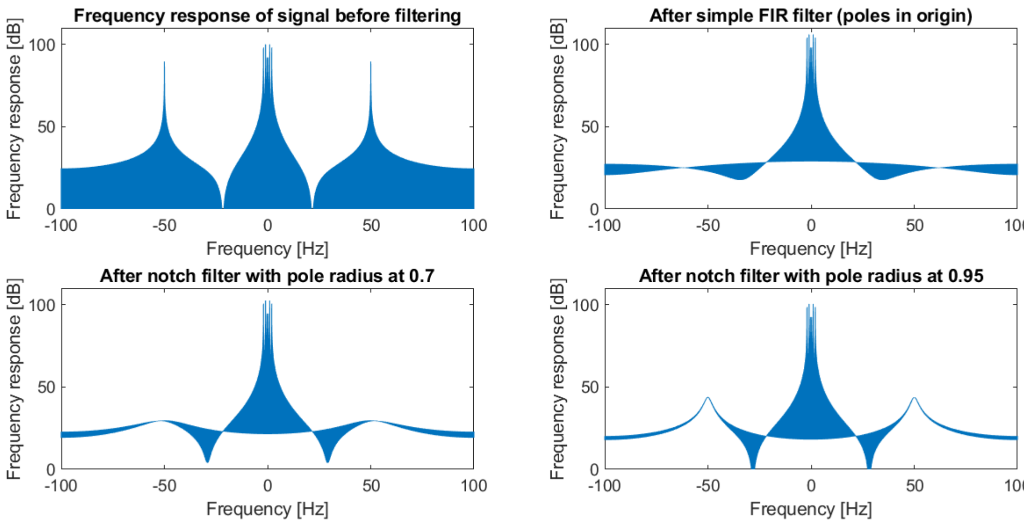
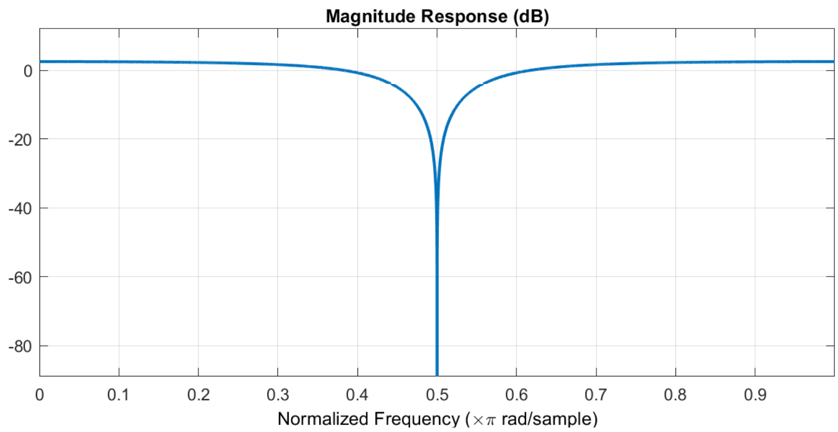


Figur 5: Frequency analysis of the blood pressure measurements after filtering with three different notch filters. The three filters have the poles located in the origin and at radius 0.7 and 0.95.



Figur 6: Magnitude response for notch filter, shown for positive normalized frequencies. The Figure is plotted in MATLAB, which normalizes frequency wrt the Nyquist frequency instead of the sampling frequency. The magnitude drops drastically at the 50 Hz noise, which has MATLAB-normalized frequency of $f = \frac{50}{200/2} = 0.5$.

zeros. The two poles are in other words located in the origin. The transfer function of this filter is given as:

$$H(z) = \frac{(z-z_1)(z-z_2)}{z^2} = \frac{(z-j)(z+j)}{z^2} = \underline{\frac{z^2+1}{z^2}}$$

$$H(z) = \frac{(1-e^{j\frac{\pi}{2}}z^{-1})(1-e^{-j\frac{\pi}{2}}z^{-1})}{(1-0z^{-1})^2} = (1-2\cos(\frac{\pi}{2})z^{-1}+z^{-2}) = \underline{1+z^{-2}}$$

The pole-zero plot and the magnitude response of the simple FIR filter is given in Figures 7 and 8. This simple FIR filter works well for this example since we don't need the poles because there is

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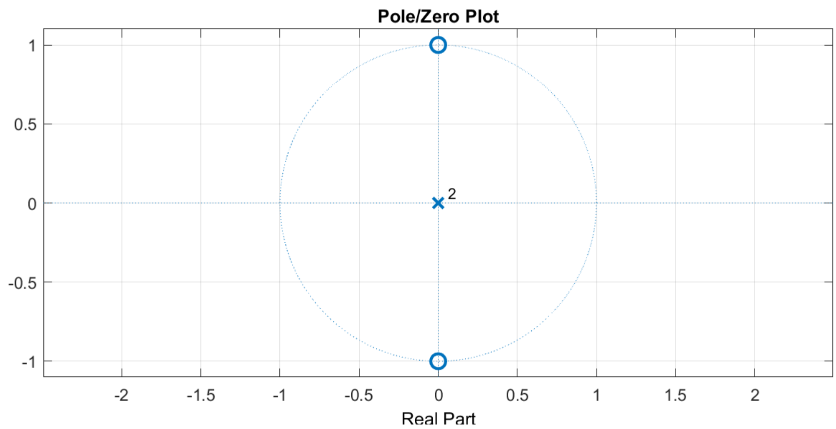
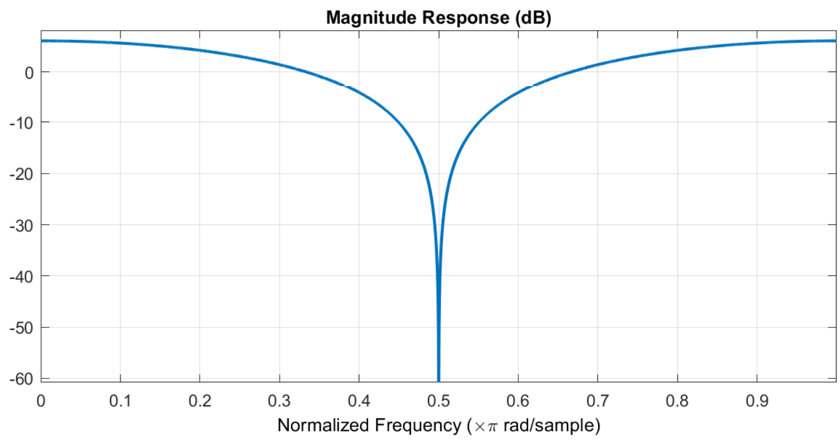


Figur 7: Pole-zero plot for simple FIR filter

such a large angular distance to the other dominant frequencies that suppression of them will be negligible.



Figur 8: Magnitude response for simple FIR filter - for illustration. The Figure is plotted in MATLAB, which normalizes frequency wrt the Nyquist frequency instead of the sampling frequency. The magnitude drops drastically at the 50 Hz noise, which has MATLAB-normalized frequency of $f = \frac{50}{200/2} = 0.5$.

Grading: 1p. for correct H(z) and pole-zero plot. 1p. for reasoning. Reasoning should include comment about not needing poles because of large angular distance to the other frequencies. A filter with two zeros is needed to remove both ±50 Hz frequencies.

Svar:

Forslag til fasit, versjon-01:

(Fortsettes på side 8.)

Oppgave 2 Z-transform (6 p.)

a) Hva er z-transformen til følgende sekvens? Hva er konvergensområdet?

$$x[n] = \left(\frac{4}{3}\right)^n u[1-n]$$

2 p.

b) Et stabilt system har følgende poler og nullpunkter:

$$z_1 = j, \quad z_2 = -j, \quad p_1 = -\frac{1}{2} + j\frac{1}{2}, \quad p_2 = -\frac{1}{2} - j\frac{1}{2}$$

Det er kjent at frekvensresponsen $H(e^{j\omega})$ til systemet ved $\omega = 0$ er lik 0.6, mao $H(e^{j0}) = 0.6$.

- Hva er konvergensområdet? Begrunn svaret. 2 p.
- Finn systemfunksjonen $H(z)$. Vis utregning. 2 p.

Svar:

a) The sequence x[n] begins at $n = 1$ and continues (backwards in time) to $-\infty$. The z-transform is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{-\infty}^{1} \left(\frac{4}{3}\right)^n z^{-n} = \sum_{-1}^{\infty} \left(\frac{3z}{4}\right)^n = \left(\frac{3z}{4}\right)^{-1} \sum_{0}^{\infty} \left(\frac{3z}{4}\right)^n$$

$$\left(or \sum_{-1}^{\infty} \left(\frac{3z}{4}\right)^n = \sum_{0}^{\infty} \left(\frac{3z}{4}\right)^n + \left(\frac{3z}{4}\right)^{-1} = \ldots\right)$$

$$= \left(\frac{4}{3z}\right)\left(\frac{1}{1-(3z/4)}\right) = \underline{\frac{-16/9}{z(z-4/3)} = \frac{4/3}{z(1-(3/4)z)}}, \qquad |z| < \frac{4}{3}$$

and region of convergence (ROC) is thus $|z| < \frac{4}{3}$.
Grading: 1p. for correct z-transform answer and 1p for correct ROC.

b)

- The ROC cannot include any poles. Since we know that the system is stable, the ROC must include the unit circle. We therefore have a causal system where the ROC extends outward from the outermost pole. Here the distance to both poles is $|0.5 \pm j0.5| = 1/\sqrt{2}$. The ROC is therefore

$$|z| > \frac{1}{\sqrt{2}}$$

Grading: 1p. for correct reasoning including deducing it is causal, and 1p for correct ROC equation.

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- The z-transform is

$$H(z) = C\frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)} = C\frac{(z-j)(z+j)}{(z+0.5-j0.5)(z+0.5+j0.5)}$$
$$= C\frac{z^2+1}{z^2+z+0.5}$$

Since we know that $H(e^{j\omega}) = 0.6$ for $\omega = 0$, we know that $H(z) = 0.6$ for $z = e^{j0} = 1$.

$$H(z=1) = C\frac{1^2+1}{1^2+1+0.5} = C\frac{2}{2.5} = C\frac{4}{5} = 0.6$$
$$\Rightarrow C = \frac{3}{4}$$

We thus get the following system function:

$$H(z) = \frac{\frac{3}{4}(z^2+1)}{z^2+z+0.5}$$

Grading: 1p. for correct equation setup and 1p. for correct answer including correct constant C value.

Svar:

Forslag til fasit, versjon-01:

Oppgave 3 Sampling (9 p.)

a) Du har et måleinstrument med samplingsrate på 2000 Hz.

- Hva er foldefrekvensen og hva slags betydning har den? 1 p.
- Du ønsker å analysere et kontinuerlig 500 Hz-cosinussignal, $x_c(t)$. Hva er perioden T til dette signalet? I følge Shannon-kriteriet, hva bør samplingsfrekvensen minst være for dette signalet? 1 p.
- Du sampler dette signalet med måleinstrumentet ditt. Hva blir uttrykket for ditt samplede signal $x[n]$? Du kan anta ingen kvantiseringsfeil og at fasen $\phi = 0$ ved tid $t = 0$. Er $x[n]$ periodisk? Begrunn svaret. 2 p.

b)
- Tegn frekvensresponsen $X_c(e^{j2\pi F})$ av det kontinuerlige signalet $x_c(t)$ gitt i a). Ha tydelige aksebenevninger for både x- og y-aksen. La x-aksen for begge plott være frekvensområdet $[-6000, 6000]$ Hz. 1 p.
- I et nytt plott, tegn frekvensresponsen til ditt samplede signal $x[n]$. Ha tydelige aksebenevninger for både x- og y-aksen. La x-aksen for begge plott være frekvensområdet $[-6000, 6000]$ Hz. 1 p.
- Kommenter forskjeller i plottene og forklar hvorfor de oppstår. 1 p.

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c) Hvilken frekvens vil du registrere på måleinstrumentet ditt om du analyserer et signal som har frekvens på 1500 Hz? Hva om signalet har frekvens på 4300 Hz? Begrunn svarene. 2 p.

Svar:

a)
- The folding frequency is half the sampling frequency, i.e. $F_{fold} = F_s/2 = 1000$ Hz. All frequencies above this will be folded, according to the sampling theorem. The Nyquist frequency is the same as the folding frequency.
  Grading: Correct answer but insufficient explanation gives 0.5p.
- The period, $T = \frac{1}{F_0} = \frac{1}{500Hz} = \underline{2\text{ ms}}$. The sampling frequency should be larger than $2F_{max} = 2 * 500Hz = 1000Hz$, i.e. this signal requires $\underline{F_s > 1000 \text{ Hz}}$.
  Grading: 0.5p given for correct answer to each question.
- The sampled signal is

  $$\underline{x[n] = x_c(nT)} = cos(2\pi F_0 nT) = cos(2\pi \frac{F_0}{F_s} n) = \underline{cos\left(\frac{\pi n}{2}\right)}$$

  The sampled signal is periodic as $x[n+N] = x[n]$ with period $N = 4$.
  Grading: Correct answer for $x[n]$ gives 1p. Only having $x[n] = x_c(nT)$ gives 0.5p. 1 point for correct reasoning on periodicity (only yes/no answer gives no credit).

b)
- The frequency response $X_c(e^{j2\pi F})$ should be plotted as two stems of coefficient value $\frac{1}{2}$ and location on the x-axis at $F = \pm 500$ Hz. This falls directly from the Euler identity of a cosine: $x_c(t) = \cos(\Omega_0 t) = \cos(2\pi F_0 t) = \frac{1}{2}(e^{j2\pi F_0 t} + e^{-j2\pi F_0 t})$.
- Sampling in time results in periodization in frequency space. You get scaled copies of $X_c(e^{j2\pi F})$ with placement depending on the sampling frequency.

  $$X(e^{j2\pi FT}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j2\pi(F - kF_s))$$

  The frequency response $X(e^{j2\pi FT})$ should therefore be plotted as repeated AND scaled spectra of $X_c(e^{j2\pi F})$ centered at every multiple of F_s. The scaling is also dependent on the applied sampling, and the coefficient values are now scaled to $\frac{1}{2T}$. We therefore get $\underline{X(e^{j2\pi FT}) = \frac{1}{2T}}$ at the following frequencies: $\underline{F = kF_s \pm F_0 = \{\pm 500, \pm 1500, \pm 2500, \pm 3500, \pm 4500, \pm 5500\} \text{ Hz}}$.

Grading: This question can be answered without having the exact expressions for $x[n]$ from a) since the student should know what the

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frequency response for a continuous cosine is without calculating it from the CTFT-equation. The plot of $X_c(e^{j2\pi F})$ gives 1p.
The plot of $X(e^{j2\pi FT})$ can also be drawn without any prior calculations. To get full 2p.-score on the plot of $X(e^{j2\pi FT})$ and description of differences, both the $\frac{1}{T}$-scaling and the exact locations of the spectra replica must be included AND EXPLAINED.

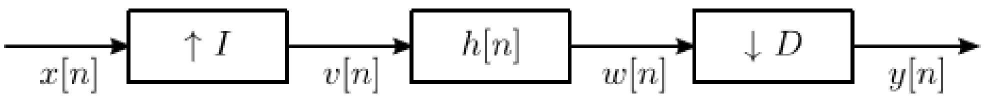
c)
- The 1500 Hz signal will be folded once and aliased to the apparent frequency of $F_a = |F_0 - F_s| = |1500 - 2000| = 500$ Hz.
- The 2300 Hz signal will be folded and aliased to the apparent frequency of $F_a = |F_0 - 2F_s| = |4300 - 2 * 2000| = 300$ Hz.

Grading: 1p. per correct answer with reasoning. Reasoning can also be a sketch of the sampled signal's frequency response, i.e. a figure similar to the figure in b) with the comment that the apparent frequencies will be the location of the repeated spectra within the $[-F_s/2, F_s/2]$ range.

Svar:

Forslag til fasit, versjon-01:

Oppgave 4 Samplingsratekonvertering (13 p.)



Figur 9: System for samplingsratekonvertering.

Figur 9 viser et standard system for ratekonvertering (opp- og nedsampling) av diskrete signaler. Anta at signalet $x[n]$ er samplet fra et kontinuerlig signal $x_c(t)$ med samplingsfrekvens $F_s = 10$ kHz. I figuren betyr I interpolasjonsfaktoren og D desimeringsfaktoren. Filteret $h[n]$ er et ideelt lavpassfilter med knekkfrekvens ω_c.

a) Hva er funksjonen/formålet til dette lavpassfilteret? Hint: Det har *to* funksjoner. 2 p.

b) Uttrykk knekkfrekvensen, ω_c, til lavpassfilteret ved hjelp av I og D. Her er ω_c digital vinkelfrekvens. 1 p.

c) Anta at knekkfrekvensen er $\omega_c = \pi/3$ og at magnituderesponsen til inngangssignalet $|X(e^{j\omega})|$ er som i figur 10. Skisser magnituderesponsen $|V(e^{j\omega})|$, $|W(e^{j\omega})|$ og $|Y(e^{j\omega})|$ for

- $I = 3,\ D = 2$ 4 p.
- $I = 2,\ D = 3$ 4 p.

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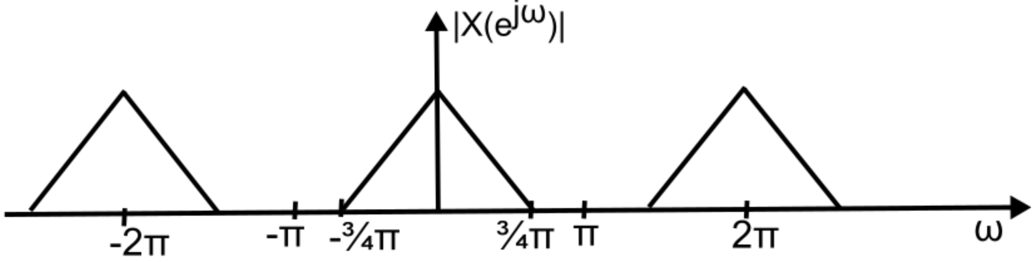

Figur 10: Magnituderesponsen til inngangssignalet.

Bruk samme frekvensakse og indiker senterfrekvenser og bredder for alle spektra, som i figur 10. NB: Du får lov å anta at enhver endring av maksimal magnitude i løpet av prosessen er kompensert av forsterkningen i lavpassfilteret. Du kan derfor skissere slik at alle responsene har samme konstante maksimal magnitude.

d) Anta at lavpassfilteret er korrekt satt opp i henhold til b). Finn det høyeste forholdet D/I som fortsatt muliggjør perfekt rekonstruksjon av $x[n]$ fra $y[n]$. Begrunn svaret. 2 p.

Svar:

a) This filter aims to:

- Remove «images» after up-sampling/interpolation. See figure 12.10 in Manolakis. These images are a consequence of the insertion of $I-1$ zeros between the samples of $x[n]$. (1 point)
- The filter is also an anti-aliasing filter before the decimation. See for instance figure 12.6 in Manolakis. (1 point)

The specifications of the filter is then the «strictest» of the specifications for the corresponsing anti-images and anti-aliasing filters.

b) Decimation using a factor D, gives us a discrete-time sample rate of $\omega_s = 2\pi/D$. This means aliasing would occur for frequencies above $\omega_s/2 = \pi/D$. Similarly, an interpolation factor I causes images outside π/I. The filter thus has to cut at the most strict of these criteria, meaning $\omega_c = min\{\pi/D, \pi/I\}$.

c)
- Case $I = 3, D = 2$. See figure 11. For the $|V(e^{j\omega})|$, the upsampling causes the original spectrum to be compressed by the factor $I = 3$. The original $-\pi \to \pi$ interval compresses into $-\pi/3 \to \pi/3$. The maximum frequency of the signal maps to $\pi/4$. The upsampling causes $I-1 = 2$ images centered at frequencies $\pm 2\pi/3$ and $\pm 4\pi/3$. The half width of all spectra is $\pi/4$. The guard band between the spectra is $2(\pi/3 - \pi/4) = \pi/6$.

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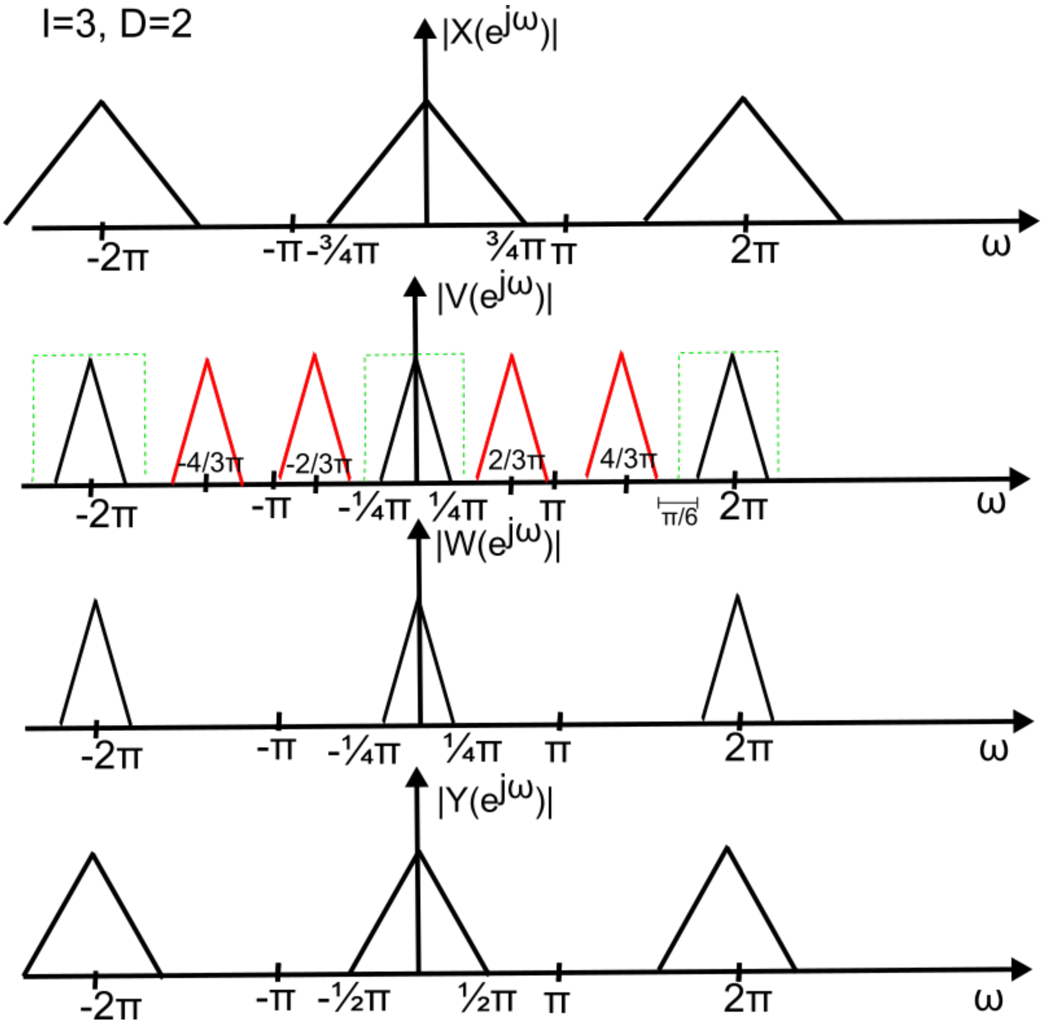


Figur 11: Magnitude responses for $I = 3, D = 2$

The lowpass filter removes the images without affecting the original signal as $2\pi/3 - \pi/4 > \pi/3$.

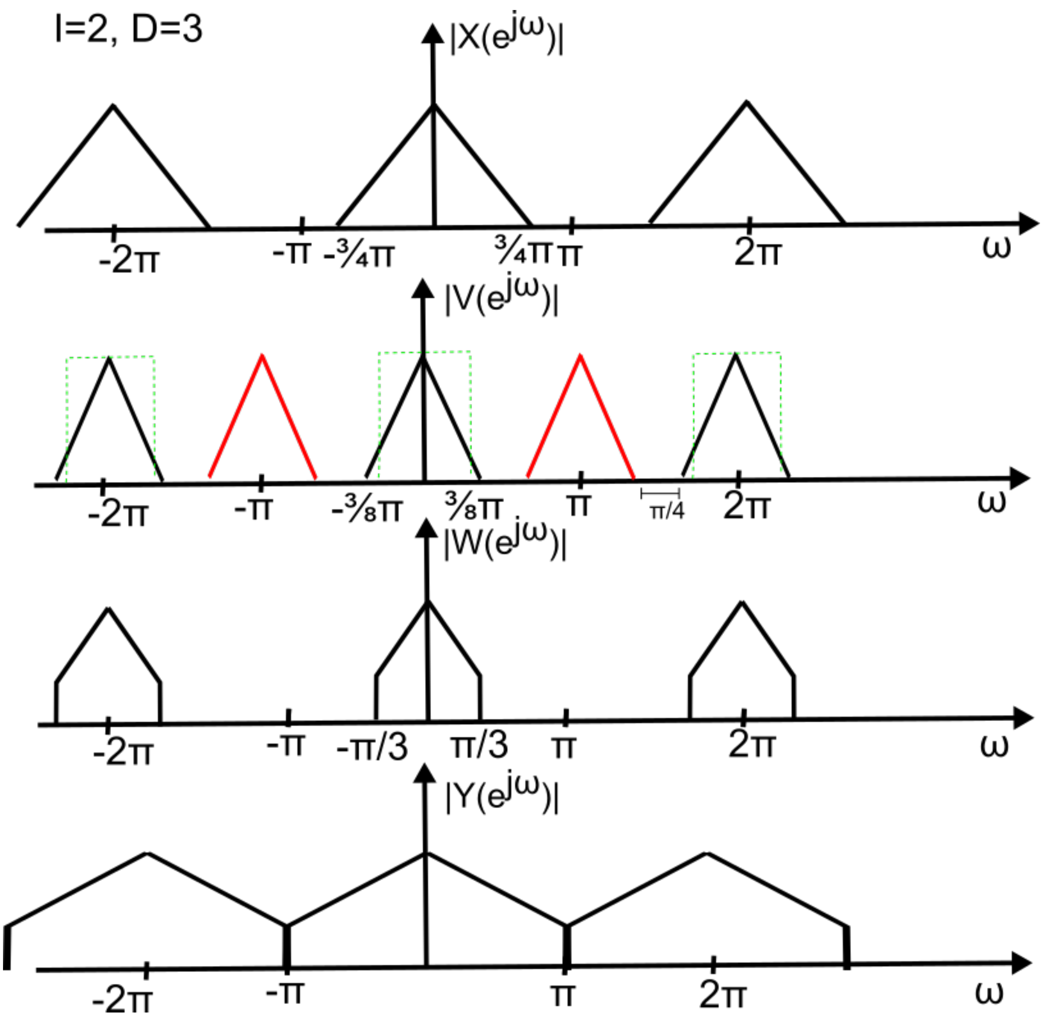
Finally, the decimation with factor $D = 2$ stretches the spectra to have widths $\pi/2$.

- Case $I = 2, D = 3$. See figure 12. For the $|V(e^{j\omega})|$, the upsampling causes the original spectrum to be compressed by the factor $I = 2$. The original $-\pi \to \pi$ interval compresses into $-\pi/2 \to \pi/2$. The maximum frequency of the signal maps to $3\pi/8$. The upsampling causes $I - 1 = 1$ image centered at frequency $\pm\pi$. The half width of all spectra is $3\pi/8$. The guard band between the spectra is $\pi/4$.

  Note that the maximum frequency of the signal is higher than the cutoff frequency of the lowpass filter, cutting at $\pi/3$, meaning that the lowpass filter removes the images but, acting as an anti-aliasing filter, *also* removes a bit of the original signal.

  Finally, the decimation with factor $D = 2$ stretches the spectra, so that the maximum frequency of the signal (now $\pi/3$) is stretched

(Fortsettes på side 14.)



Figur 12: Magnitude responses for $I = 2, D = 3$

to π.

Grading: For the spectra after interpolation: 1 point to get the width and 1 point to get the images correctly centered. Then, 1 point for correct lowpass filtered spectra. Finally, 1 point for correct spectrum expansion after decimation.

d) The largest D/I ratio is the ratio which causes the highest frequency component of the signal to end up at $\pm\pi$, without the lowpass filter removing anything of the original signal. This ratio can be found in many different ways. If you managed c), it can easily be found by inspecting the figures for $I = 3$ and $D = 2$. One may observe that the highest frequency component ended up at $\pi/2$, meaning we could tolerate twice as high downsampling factor before the antialiasing lowpass filter would have to remove parts of the original signal. The ratio then becomes $D/I = 4/3$. The cutoff filter of the lowpass filter would be $\pi/4$, which is exactly where the highest frequency component of the original signal ends up after interpolation with $I = 3$, as seen

(Fortsettes på side 15.)

in c). One could also have found it without solving c) at all, by inspecting the physical frequencies. The highest frequency of the signal is $F = (3\pi/4)/(2\pi)F_s = 3/8F_s$. This means it has to be sampled at $F_{s,min} = 2 * F = 3/4F_s$, which implies a supported resampling rate of $F_{s,min}/F_s = I/D = 3/4$.
Grading: 1 point to get the ratio. 1 point for a meaningful explanation.

Svar:

Forslag til fasit, versjon-01:

Oppgave 5 FIR filter design (10 p.)

a) Gitt frekvensresponsen til et ideelt lavpassfilter, $H_{LP}(e^{j\omega})$, med kuttfrekvens ω_c, vis at impulsresponsen blir

$$h_{LP}[n] = \frac{sin(\omega_c n)}{\pi n}, -\infty < n < \infty$$

1 p.

b) Forklar hvorfor dette filteret ikke lar seg implementere i praksis. 1 p.

c) En måte å tilnærme det ideelle filteret med et FIR filter, er ved å bruke en reell vindusfunksjon for å velge ut en del av den ideelle impulsresponsen. Benevn vindusfunksjonen som $w[n]$ og anta at det har lengde $2M + 1$ og er sentrert om $n = 0$. Forklar kvalitativt (med ord) hvordan DTFT av vindusfunksjonen påvirker egenskapene til lavpassfilteret. 2 p.

d) Du har funnet ut for din anvendelse må transisjonsbåndet være mindre enn 1 kHz. Samplingsfrekvensen er 10 kHz. Du har slått opp i tabell og funnet at det rektangulære vinduet har oppgitt $\Delta\omega = 4\pi/L$ hvor $\Delta\omega$ er transisjonsbåndet og L er vinduets lengde.

- Hvilken filterorden må du velge? 1 p.
- Vil filteret ditt ha lineær fase? Begrunn svaret. 1 p.
- Ut fra det som er gitt i oppgaven til nå, kan du avgjøre hvorvidt filteret ditt vil bli FIR type I, II, III eller IV? Begrunn svaret. 1 p.

e) Det viser seg at med filteret du kommer fram til, har for mye passbåndrippel. Du prøver å løse dette ved å øke filterorden betraktelig. Men det hjelper ingenting. Forklar hvorfor det ikke hjelper og foreslå en annnen løsning på dette problemet. 1 p.

f) Du må gi opp å designe filteret ved hjelp av vinduer, da filtere som oppfyller krav til transisjonsbånd blir altfor lange. Hva slags alternative designstrategier vil du foreslå i stedet? Forklar kort (maks. 100 ord) fordeler/ulemper med de ulike fremgangsmåtene og hvilke vurderinger du må gjøre. 2 p.

(Fortsettes på side 16.)

Svar:

a) The frequency response of the ideal lowpass filter is written as

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

Using the DTFT synthesis equation from the formula sheet:

$$h_{LP}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$$
$$h_{LP}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega$$
$$h_{LP}[n] = \frac{1}{2\pi nj}\left[e^{j\omega n}\right]_{-\omega_c}^{\omega_c}$$
$$h_{LP}[n] = \frac{1}{2\pi nj}\left[e^{j\omega_c n} - e^{-j\omega_c n}\right]$$
$$h_{LP}[n] = \frac{1}{\pi n} sin(\omega_c n)$$

In the last step, we used the Euler formula for the sine function (also in formula sheet).

b) This filter is not practically realizable. $h_{LP}[n]$ is not absolutely summable, $\sum_{n=-\infty}^{\infty} |h_{LP}[n]| = \infty$, meaning it per definition is an unstable system. This also means it has no z-transform or frequency response.

c) The DTFT of a window function, at least the symmetric ones that we have considered in our course, typically have a sinc-like shape. The main lobe of the DTFT affects the transition band, while the side lobe level sets the pass- and stop band ripple levels. The length of the window function directly affects the width of the main lobe, meaning a long filter will have a narrow main lobe, which in turn means a short transition band.
Grading: 1 Point for explaining main lobe. 1 Point for the side lobe.

d)
- First we have to convert the transition band to ω

$$\Delta\omega = 2\pi f = 2\pi\Delta F/F_s = 2\pi 1/10 = \pi/5 \quad (1)$$

Inserting this into the equation for filter length yields:

$$L = \frac{4\pi}{\Delta\omega} = \frac{4\pi}{\pi/5} = 20 \quad (2)$$

This means the filter order is $N = L - 1 = 19$. You would also get credit here by answering $N = 20$, but then it must be clearly explained that you do this in order to get a type I FIR filter (even order).

(Fortsettes på side 17.)

- Yes, as long as the window is symmetric, so will the FIR coefficients. Symmetric coefficients implies linear phase. NB! No explanation -> No credits
- Yes, the information indicates a FIR II filter, unless you explicitly chose to bump the filter length by 1 to get type I in d). NB! No explanation -> No credits

e) The ripple is defined by the window side lobe level, which for the fixed windows, is independent of the window length. The only thing you can do to reduce the ripple levels in a fixed window design, is to select a different window. Another alternative would be to use an adjustable window, such as the Kaiser window.

f) Here the student is given some flexibility to show what knowledge he/she has gained related to FIR/IIR filtering. Some key points that could be mentioned:

- Optimal FIR filter design by Parks-McClellan/Remez Exchange algorithm. Trade order for equiripple response. This is the best solution of the ones with linear phase.
- If linear phase can be omitted -> IIR filter
- IIR filters generally have lower order for same specifications
- IIR filters -> Impossible to have perfect linear phase
- IIR filters -> Have poles. Need to make sure stability
- Butterworth IIR filter -> Maximally flat response. Filter order still too high? Advantage: Almost constant group delay in passband.
- Chebyshev I -> Trade filter order for equiripple response in passband. More variations in group delay than Butterworth.
- Chebyshev II -> Trade filter order for equiripple response in stopband. More variations in group delay than Butterworth, but less than Chebyshev I.
- Elliptic filter -> Trade filter order for equiripple response in both passband and stopbands. Large variations in group delay.
- Choice of filter will be dependent on phase / group delay requirements.
- IIR filters designed in continuous domain. Need to verify the specifications are fulfilled also after discretization (advanced)

<u>Grading:</u> 1 point is given for presenting alternatives. 1 point is given for discussing advantages/disadvantages/tradeoffs.

(Fortsettes på side 18.)

Formelsamling

Grunnleggende sammenhenger:

$$\begin{aligned}
\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\
\cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\
\sin 2\alpha &= 2\sin\alpha\cos\alpha \\
\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\
\sin\alpha + \sin\beta &= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\sin\alpha - \sin\beta &= 2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\
\cos\alpha + \cos\beta &= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\cos\alpha - \cos\beta &= -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\
\cos^2\alpha + \sin^2\alpha &= 1 \\
\cos\alpha &= \frac{1}{2}(e^{j\alpha} + e^{-j\alpha}) \\
\sin\alpha &= \frac{1}{2j}(e^{j\alpha} - e^{-j\alpha}) \\
\sum_{n=0}^{N-1} a^n &= \begin{cases} N & \text{for } a = 1 \\ \frac{1-a^N}{1-a} & \text{ellers} \end{cases} \\
ax^2 + bx + c = 0 &\Leftrightarrow x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

Lineær konvolusjon:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] = h[n] * x[n]$$

Sirkulær konvolusjon:

$$y[n] = x[n] \textcircled{N} h[n] = \sum_{k=0}^{N-1} x[k]h[< n-k >_N] = \sum_{k=0}^{N-1} x[< n-k >_N]h[k] = h[n] \textcircled{N} x[n]$$

Kontinuerlig-tid-fouriertransformasjon (CTFT):

$$\begin{aligned}
\text{Analyse:} \quad X(j\Omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \\
\text{Syntese:} \quad x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega
\end{aligned}$$

Kontinuerlig-tid-fourierrekke (CTFS):

$$\begin{aligned}
\text{Analyse:} \quad c_k &= \frac{1}{T_o}\int_{T_0} x(t)e^{-jk\Omega_0 t}dt \\
\text{Syntese:} \quad x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t}
\end{aligned}$$

(Fortsettes på side 19.)

Diskret tid-fourierrekke (DTFS):

$$\begin{aligned} \text{Analyse:} \qquad c_k &= \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn} \\ \text{Syntese:} \qquad x[n] &= \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn} \end{aligned}$$

Diskret tid-fouriertransformasjon (DTFT):

$$\begin{aligned} \text{Analyse:} \qquad X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \\ \text{Syntese:} \qquad x[n] &= \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \end{aligned}$$

Diskret fouriertransformasjon (DFT):

$$\begin{aligned} \text{Analyse:} \qquad X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}, \quad 0 \le k \le N-1 \\ \text{Syntese:} \qquad x[n] &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N}, \quad 0 \le n \le N-1 \end{aligned}$$

z-transformasjonen:

$$\text{Analyse:} \qquad X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$