Mathematical foundations

Read the relevant lecure slides.

1 Sets

1.1 Exercise

What is the difference between $\emptyset$ and $\{\emptyset\}$?

1.1.1 Solution

$\emptyset$ is the empty set, i.e, the set with no elements. $\{\emptyset\}$ is the set containing one element, the empty set.

1.2 Exercise

In this exercise we will use the following sets:

- $A = \{a, b, c, d\}$
- $B = \{d, f, e, r, k\}$
- $C = \{r, e, m\}$
- $D = \{q, l\}$
- $E = \{\}$
- Δ is the universal set.

What is the cardinality of each of these sets?

List all the elements in the following sets:

1. $A \cup B$.
2. $A \cup (B \cap C)$.
3. $(A \cap B) \cup (C \cap A)$.
4. $B \setminus C$.
5. $C \setminus B$.
6. $D \cap \overline{E}$.
7. $D \cup \overline{E}$.

1.2.1 Solution

Cardinalities:

1. $|A| = 4.$
2. $|B| = 5$
3. $|C| = 3.$
4. $|D| = 2.$
5. $|E| = |\emptyset| = 0.$

Sets:

1. $A \cup B = \{a, b, c, d, e, f, k, r\}$
2. $A \cup (B \cap C) = A \cup \{e, r\} = \{a, b, c, d, e, r\}.$
3. $(A \cap B) \cup (C \cap A) = \{d\} \cup \emptyset = \{d\}$
4. $B \setminus C = \{d, f, k\}.$
5. $C \setminus B = \{m\}.$
6. $D \cap \overline{E} = D \cap \Delta = D = \{q, l\}.$
7. $D \cup \overline{E} = D \cup \Delta = \Delta.$

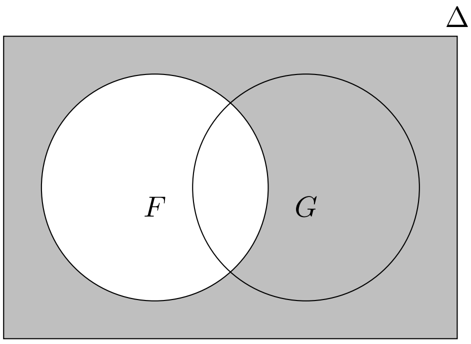
1.3 Exercise

Let F and G be two arbitrary sets and Δ the universal set. Draw Venn diagrams containing the sets F, G and Δ and shade the area representing the following sets:

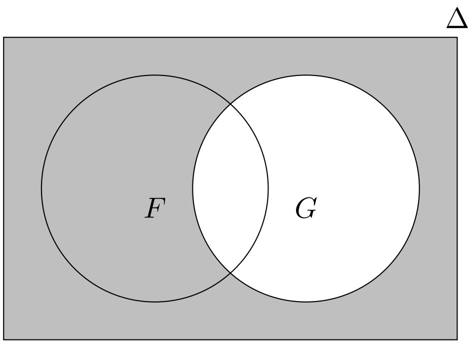
1. $\overline{F}.$
2. $\overline{G}.$
3. $\overline{(F \cup G)}.$
4. $\overline{F} \cap \overline{G}.$
5. $\overline{(F \cap G)}.$
6. $\overline{F} \cup \overline{G}.$

1.3.1 Solution

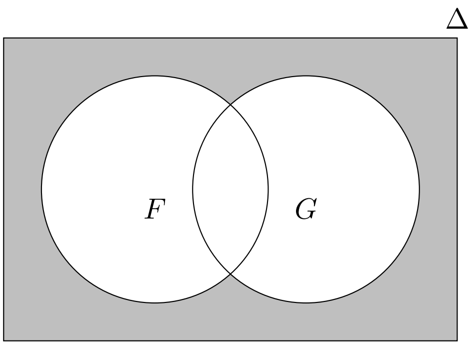
1. Exercise 1.



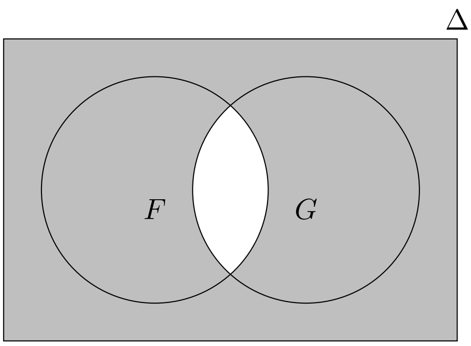
2. Exercise 2.



3. Exercise 3 and 4.



4. Exercise 5 and 6.



1.4 Exercise

Create three sets A, B and C such that the following hold:

- The union of A and B is $\{1, 2, 3, 4\}$.
- The intersection of A and C is $\{3\}$.

- The union of B and C is $\{3, 4, 5, 6\}$.
- The intersection of B and C is $\{4\}$.

1.4.1 Solution

- $A = \{1, 2, 3\}$
- $B = \{4\}$
- $C = \{3, 4, 5, 6\}$

1.5 Exercise

Let $A = \{1, 2, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$ and decide if the following hold

- $1 \in A$
- $2 \in A$
- $3 \in A$
- $\emptyset \in A$
- $\{1\} \in A$
- $\{1, 3\} \in A$
- $\{1, 2, \{1, 2\}\} \in A$
- $\emptyset \subseteq A$
- $\{1\} \subseteq A$
- $\{1, 3\} \subseteq A$
- $\{1, 2, \{1, 2\}\} \subseteq A$
- $\{\{1, 2, 3\}\} \in A$

1.5.1 Solution

- $1 \in A$ true
- $2 \in A$ true
- $3 \in A$ false
- $\emptyset \in A$ false
- $\{1\} \in A$ false
- $\{1, 3\} \in A$ true
- $\{1, 2, \{1, 2\}\} \in A$ false
- $\emptyset \subseteq A$ true
- $\{1\} \subseteq A$ true

- $\{1, 3\} \subseteq A$ false
- $\{1, 2, \{1, 2\}\} \subseteq A$ true
- $\{\{1, 2, 3\}\} \in A$ false

2 Relations

2.1 Exercise

Let A be the set $A = \{a, b, c, d, e, f\}$. Create non-empty relations R_i on A such that the conditions below hold.

1. $R_1 = A \times A$
2. R_2 is reflexive.
3. R_3 is symmetric.
4. R_4 is transitive.
5. R_5 is irreflexive.

2.1.1 Solution

There is only one solution to R_1 and R_2. There are many solutions to R_3, R_4 and R_5.

$$
\begin{aligned}
R_1 = \quad & \{ \\
& \langle a,a\rangle, \langle a,b\rangle, \langle a,c\rangle, \langle a,d\rangle, \langle a,e\rangle, \langle a,f\rangle, \\
& \langle b,a\rangle, \langle b,b\rangle, \langle b,c\rangle, \langle b,d\rangle, \langle b,e\rangle, \langle b,f\rangle, \\
& \langle c,a\rangle, \langle c,b\rangle, \langle c,c\rangle, \langle c,d\rangle, \langle c,e\rangle, \langle c,f\rangle, \\
& \langle d,a\rangle, \langle d,b\rangle, \langle d,c\rangle, \langle d,d\rangle, \langle d,e\rangle, \langle d,f\rangle, \\
& \langle e,a\rangle, \langle e,b\rangle, \langle e,c\rangle, \langle e,d\rangle, \langle e,e\rangle, \langle e,f\rangle, \\
& \langle f,a\rangle, \langle f,b\rangle, \langle f,c\rangle, \langle f,d\rangle, \langle f,e\rangle, \langle f,f\rangle \\
& \}
\end{aligned}
$$

$R_2 = \{\langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle, \langle d,d\rangle, \langle e,e\rangle, \langle f,f\rangle\}$

$R_3 = \{\langle a,a\rangle, \langle a,b\rangle, \langle b,a\rangle, \langle d,c\rangle, \langle c,d\rangle, \langle f,f\rangle\}$

$R_4 = \{\langle a,a\rangle, \langle a,b\rangle, \langle b,a\rangle, \langle d,c\rangle, \langle c,d\rangle, \langle f,f\rangle, \langle b,b\rangle, \langle d,d\rangle, \langle c,c\rangle\}$

$R_5 = \{\langle a,b\rangle, \langle c,d\rangle\}$

2.2 Exercise

Assume the normal intended interpretation. Which of the following relations are reflexive, transitive and/or symmetric?

- hasSister
- hasSibling
- hasFather
- hasParent

- hasAge
- hasSpouse
- likes

2.2.1 Solution

This is one normal interpretation:

- hasSister: transitive
- hasSibling: symmetric and transitive
- hasFather:
- hasParent:
- hasAge:
- hasSpouse: symmetric (transitive?)
- likes: symmetric, reflexive?

3 Propositional logic

3.1 Exercise

Let ϕ be the propositional formula $(P \wedge Q) \vee R \to S \wedge Q$.

- Create an interpretation $\mathcal{I}_1$ such that $\mathcal{I}_1 \models \phi$.
- Create an interpretation $\mathcal{I}_2$ such that $\mathcal{I}_2 \not\models \phi$.

3.1.1 Solution

- $I_1 = \{R, S, Q\}$
- $I_2 = \{R\}$

3.2 Exercise

- Find the truth table to the formula $(P \to Q) \to P$
- Find the truth table to the formula $(P \to Q) \vee (Q \to P)$
- What is there to note about the two formulae?

3.2.1 Solution

P	Q	$(P \to Q)$	$\to$	P
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	F	F

The formula is equivalent to P.

P	Q	$(P \to Q)$	$\vee$	$(Q \to P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	T	F
F	F	T	T	T

The formula is always true, it is a tautology.

3.3 Exercise

Decide the following entailment questions. If the answer is yes, then produce a proof, e.g., a truth table, which shows why the answer is yes. If the answer is no, then produce a countermodel, i.e., an interpretation which makes the first formula true and the second false.

- Does $P \vee Q$ entail Q?
- Does $P \wedge Q$ entail $P \vee Q$?
- Does $P \to (P \to Q)$ entail Q?
- Does $P \wedge \neg P$ entail Q?

3.3.1 Solution

- Does $P \vee Q$ entail Q? No.
- Does $P \wedge Q$ entail $P \vee Q$? Yes.
- Does $P \to (P \to Q)$ entail Q? No.
- Does $P \wedge \neg P$ entail Q? Yes.