# INF3580/4580 – Semantic Technologies – Spring 2017

## Lecture 5: Mathematical Foundations

Martin Giese

13th February 2017

Department of Informatics

University of Oslo

# Mandatory exercises

- Remember: Hand-in Oblig 3 by tomorrow.
- Oblig 4 published after next lecture.

# Today's Plan

1. Basic Set Algebra
2. Pairs and Relations
3. Propositional Logic

# Outline

# Motivation

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- There are some problems with this, but it's good enough for us!

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- Sets with different elements are different:

$$\{1, 2\} \neq \{2, 3\}$$

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- $x \notin \emptyset$, whatever $x$ is!

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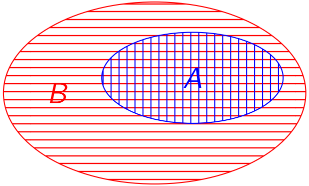

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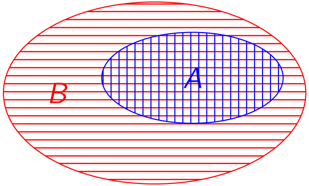

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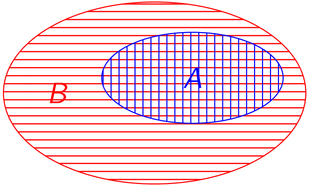


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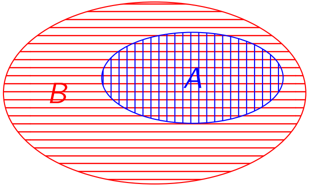


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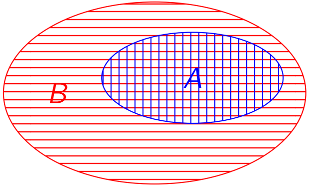


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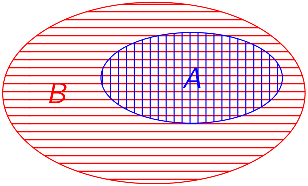


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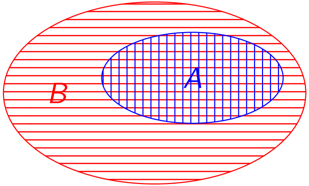


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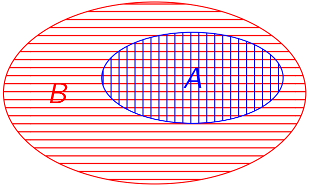


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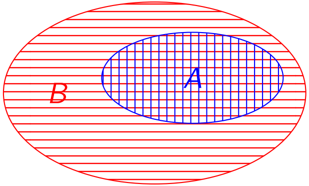


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- $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

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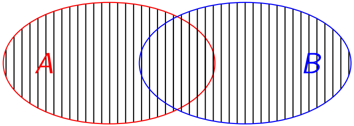


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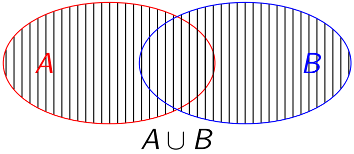

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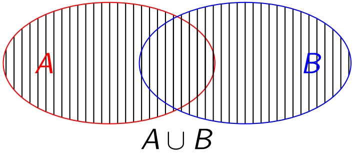

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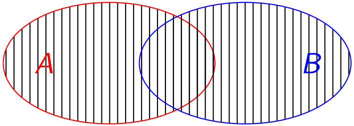


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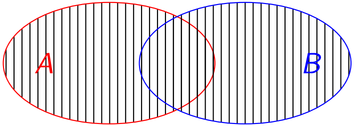


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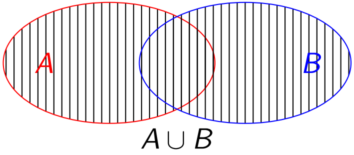


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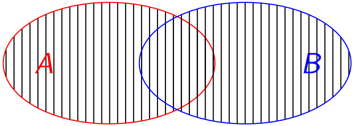


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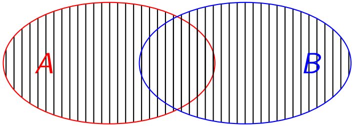


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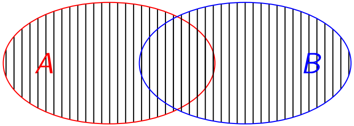


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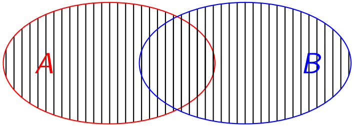


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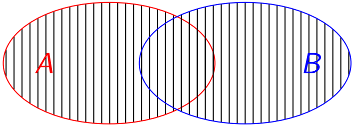


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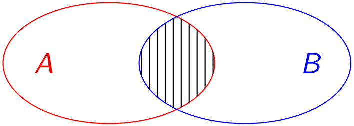


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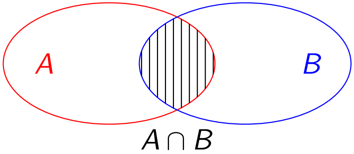

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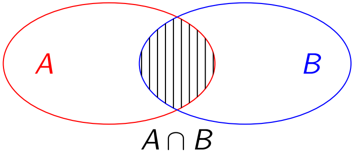

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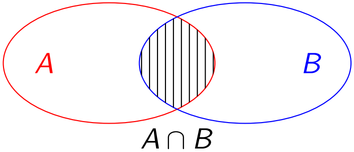

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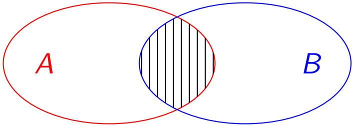


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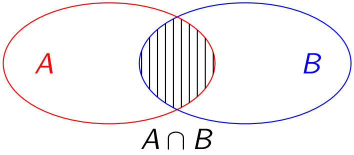


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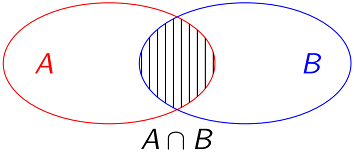


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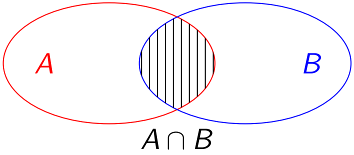


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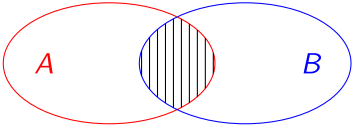


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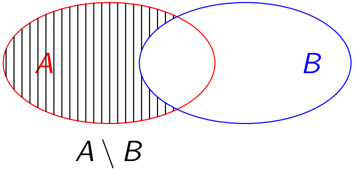

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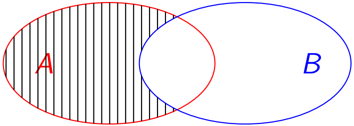


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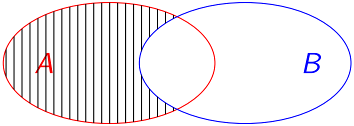


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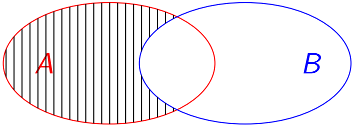


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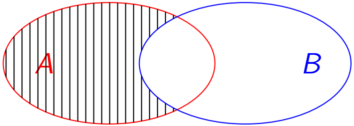


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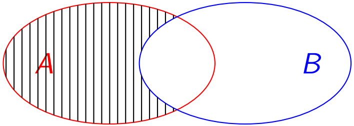


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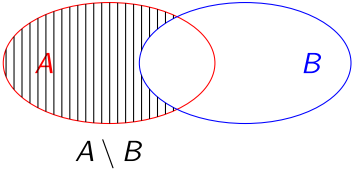

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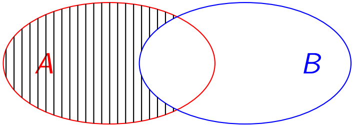


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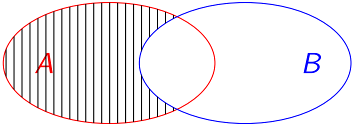


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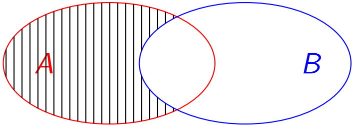


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# Question

The *symmetric difference* $A \triangle B$ of two sets contains

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Or:

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# Outline

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- Sets are good to group objects with some properties!
- How do we talk about relations between objects?

# Pairs

- A pair is an *ordered* collection of two objects

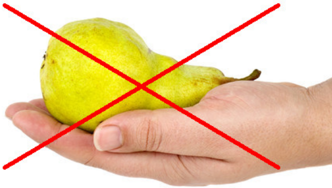

Image ©Colourbox.no

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- $\langle x, y \rangle$ is a pair, no matter if $x = y$ or not.

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  - $A = \{1, 2, 3\}$, $B = \{\text{'a'}, \text{'b'}\}$.
  - $A \times B = \{ \langle 1, \text{'a'}\rangle, \langle 2, \text{'a'}\rangle, \langle 3, \text{'a'}\rangle, \langle 1, \text{'b'}\rangle, \langle 2, \text{'b'}\rangle, \langle 3, \text{'b'}\rangle \}$
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16 relations on $A$. Generally: $2^{(|A|^2)}$

# Outline

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  - but every formula can be "parsed" uniquely.

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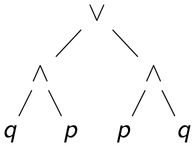

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- Let's formalize this context, a.k.a. interpretation, a.k.a. model

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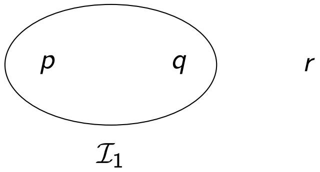

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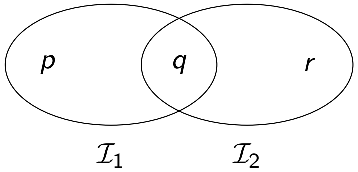

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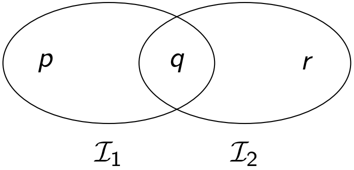


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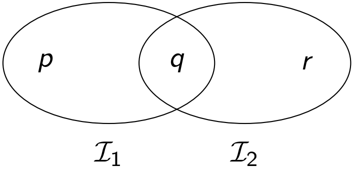


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- In other words, for all letters $p$:

$$\mathcal{I} \models p \qquad \text{if and only if} \qquad p \in \mathcal{I}$$

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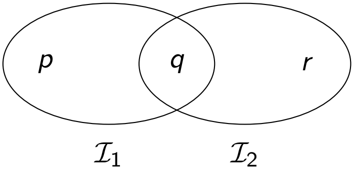


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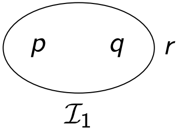
- For instance, if $\mathcal{I}_1 = \{p, q\}$:




$$\mathcal{I}_1 \not\models ((q \land r) \land (p \land q))$$

$$\mathcal{I}_1 \not\models (q \land r) \qquad \mathcal{I}_1 \models (p \land q)$$

$$\mathcal{I}_1 \models q \qquad \mathcal{I}_1 \not\models r \qquad \mathcal{I}_1 \models p \qquad \mathcal{I}_1 \models q$$

# Semantics for $\neg$, $\rightarrow$ and $\vee$

- The complete definition of $\models$ is as follows:
- For any interpretation $\mathcal{I}$, letter $p$, formulas $A, B$:
  - $\mathcal{I} \models p$ iff $p \in \mathcal{I}$
  - $\mathcal{I} \models \neg A$ iff $\mathcal{I} \not\models A$
  - $\mathcal{I} \models (A \wedge B)$ iff $\mathcal{I} \models A$ and $\mathcal{I} \models B$
  - $\mathcal{I} \models (A \vee B)$ iff $\mathcal{I} \models A$ or $\mathcal{I} \models B$ (or both)
  - $\mathcal{I} \models (A \rightarrow B)$ iff $\mathcal{I} \models A$ implies $\mathcal{I} \models B$
- Semantics of $\neg$, $\wedge$, $\vee$, $\rightarrow$ often given as *truth table*:

| $A$ | $B$ | $\neg A$ | $A \wedge B$ | $A \vee B$ | $A \rightarrow B$ |
|---|---|---|---|---|---|
| $f$ | $f$ | $t$ | $f$ | $f$ | $t$ |
| $f$ | $t$ | $t$ | $f$ | $t$ | $t$ |
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- Therefore: $(\neg p \vee (\neg q \vee (p \wedge q)))$ is a tautology!

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- Any formula is equivalent to a formula containing only the connectives $\neg$ and $\wedge$.

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