Weekly exercises on Undecidability

Exercise 1 Let $\overline{A}$ denote the *complement* of A, that is, $\overline{A} = \{w \mid w \notin A\}$.
Show that if A is a decidable language, then so is $\overline{A}$.

Exercise 2 For each language, show that it is undecidable by giving a reduction from $HALT$.

a) $L_1 = \{\langle M \rangle \mid \text{TM } M \text{ writes a \$ on the tape on every input}\}$

b) $L_2 = \{\langle M \rangle \mid \text{TM } M \text{ writes a \$ on the tape on input } 010\}$

c) $L_3 = \{\langle M \rangle \mid \text{There is no } y \text{ such that TM } M \text{ writes a \$ on the tape on input } y\}$

Exercise 3 Let $L = \{\langle M \rangle \mid \text{TM } M$ writes a \$ on the tape, during the first 100 steps of computation, on every input$\}$.
Is L decidable? Prove your answer.

Exercise 4 Let $EQUAL = \{\langle M_1, M_2 \rangle \mid L(M_1) = L(M_2)\}$.
Is $EQUAL$ decidable? If yes, show a decider for $EQUAL$, if no, show a reduction from $HALT$.

Exercise 5 Let $PI = \{k \mid$ There occurs k consecutive 0's in the decimal expansion of π $\}$.
Is PI a decidable language?

Challenge 1 Recall from the lecture Rice's theorem.

Theorem 1 (Rice's theorem) *Let R be a language consisting of Turing machine descriptions, such that R contains some, but not all Turing machine descriptions. Furthermore, let the membership in R for any Turing machine M_1, depend solely on the language of M_1, that is: $L(M_1) = L(M_2) \implies (\langle M_1 \rangle \in R \leftrightarrow \langle M_2 \rangle \in R)$. Then R is an undecidable language.*

Give a proof of Rice's theorem.
Hint: Show a reduction from $HALT$ to R. Find a yes-instance and a no-instance of R, and create a TM that recognizes the one of those language depending on whether M halts on w on not. The result from Exercise 1, may come in handy.