Weekly exercises on NP-completeness

Exercise 1 We have looked at several versions of Hamiltonicity. Here are three different problems related to the same topic:

$HP = \{\langle G \rangle \mid \text{The directed graph } G \text{ contains a Hamiltonian path}\}$.

$HC = \{\langle G \rangle \mid \text{The directed graph } G \text{ contains a Hamiltonian cycle}\}$.

$GH = \{\langle G, s, t \rangle \mid \text{The directed graph } G \text{ contains a Hamiltonian path from node } s \text{ to node } t\}$.

Show the following six reductions:

1. $HP \leq_P HC$
2. $HP \leq_P GH$
3. $HC \leq_P HP$
4. $HC \leq_P GH$
5. $GH \leq_P HP$
6. $GH \leq_P HC$

Solution proposal Solutions proposed in the group session.

Exercise 2 Let

$DOUBLE{-}SAT = \{\langle \phi \rangle \mid \text{ The Boolean formula } \phi \text{ has at least two satisfying assignments}\}$.

Show that $DOUBLE{-}SAT$ is NP-complete.

Solution proposal A certificate for $DOUBLE{-}SAT$ could be two distinct assignments, satisfying ϕ.

We reduce SAT to $DOUBLE{-}SAT$. Our reduction f takes ϕ and produces $\phi \wedge (x \vee \overline{x})$, where x is a fresh variable not found in ϕ.

If an assignment satisfies ϕ, then $\phi \wedge (x \vee \overline{x})$ has two satisfying assignments: one where we extend the assignment for ϕ with x assigned true, and another where x is assigned false.

If ϕ has no satisfying assignment, then $\phi \wedge (x \vee \overline{x})$ also has no satisfying assignment (and in particular not two).

The reduction runs in polynomial time.

Exercise 3 In the lecture, we showed that $PARTITION$ is NP-complete by reducing $SUBSET{-}SET$ to $PARTITION$. We also saw that all NP-complete problems can be reduced to each other in polynomial time.

Show that $PARTITION \leq_P SUBSET{-}SUM$.

Solution proposal The reduction f takes an instance of $PARTITION$, $\langle S \rangle$. Let us assume that $\Sigma S = k$. If k is odd, we are dealing with a no-instance of $PARTITION$, so we must produce a no-instance of $SUBSET{-}SUM$.

$\langle S, k+1 \rangle$ will do. If k is odd, f returns the instance $\langle S, \frac{k}{2} \rangle$.

If $\langle S \rangle \in PARTITION$, then it is possible to select a subset of S summing to $\frac{k}{2}$, so $\langle S, \frac{k}{2} \rangle \in SUBSET{-}SUM$ and vice versa.

Exercise 4 An *independent set* of nodes in a graph G is a subset of the nodes of G, such that no edge connects any nodes in the subset.

Let $INDEPENDENT{-}SET =$
$\{\langle G, k \rangle \mid$ The undirected graph G contains an independent set of size k or more $\}$.

Show that $INDEPENDENT{-}SET$ is NP-complete.

Solution proposal The main idea is to show that $CLIQUE \leq_P INDEPENDENT{-}SET$.

On input $\langle G, k \rangle$ the reduction returns $\langle \overline{G}, k \rangle$, where $\overline{G}$ is the *complement graph* of G, that is, G where all edges are "flipped".

A clique in G forms an independent set in $\overline{G}$.

Flipping the edges in G can be done in polynomial time.

Exercise 5 A *vertex cover* in a graph G is a subset of the nodes of G, such that any edge is connected to at least one node in the vertex cover.

Let $VERTEX{-}COVER = \{\langle G, k \rangle \mid$ The undirected graph G contains vertex cover of size k or less$\}$.

Show that $VERTEX{-}COVER$ is NP-complete.

Solution proposal One way to do this is by reducing $INDEPENDENT{-}SET$ to $VERTEX{-}COVER$.

On input $\langle G, k \rangle$, the reduction produces $\langle G, |G| - k \rangle$, where $|G|$ is the number of nodes in G.

If G has an independent set of size k, then the remaining $|G| - k$ nodes form a vertex cover. This is because any edge in G must be connected to at least one node in the cover, since no edge connects two nodes in the independent set.