Weekly exercises on P and NP

Exercise 1 Show that P is closed under

- intersection
- union.

Solution proposal Let L_1 and L_2 be languages in P. Then by definition, they have polynomial deciders, let us call them M_1 and M_2. We will show that $L_1 \cap L_2 \in P$, by constructing a polynomial decider, M. The machine M, will simulate L_1 and L_2 on the input w, and accept if both machines accept. Since polynomials are closed under addition the running time of M will be polynomial.

For union, the same suggestion works, but M will accept if any of M_1 and M_2 accept.

Exercise 2 Let $CONNECTED = \{\langle G \rangle \mid \text{The undirected graph } G \text{ is connected}\}$.

Show that $CONNECTED \in P$. Hint: look at the algorithm for deciding $PATH$ from the lecture.

Solution proposal The same algorithm as shown for $PATH$ with minor adjustments works. Since we don't have s and t, we will start in any node of G (for example the first we read from the tape), and we will accept if, in the final step, all nodes are marked.

Exercise 3 Let $TRIANGLE = \{\langle G \rangle \mid \text{The undirected graph } G \text{ contains a clique of size } 3\}$.

Show that $TRIANGLE \in P$.

Solution proposal The number of subsets of the nodes is no more than n^3, so a brute-force approach will run in polynomial time.

Exercise 4 Let $COMPOSITES = \{c \mid c = pq, \text{ for natural numbers } p, q \geq 2\}$.

Prove that $COMPOSITES \in NP$ by giving showing that $COMPOSITES$ have polynomial certificates.

Solution proposal Certificate: p and q. The verifier would then check if $p, q \geq 2$ and that $pq = c$.

Exercise 5 Let G and H be two graphs. We say that G is isomorphic to H if there exists a bijection f from the set of nodes in G to the set of nodes in H, such that u and v are neighbors in G if and only if $f(u)$ and $f(v)$ are neighbors in H.

Now we define the language $GRAPH{-}ISOMORPHISM = \{\langle G_1, G_2\rangle \mid G_1 \text{ is isomorphic to } G_2\}$. Show that $GRAPH - ISOMORPHISM \in NP$.

Solution proposal Certificate: the bijection f.

Exercise 6 We say that graph G has a *vertex cover* of size k, if there exists a subset of the nodes from G such that every edge in G touches at least one node from the subset.

Let $VERTEX{-}COVER = \{\langle G, k\rangle \mid G \text{ is an undirected graph that has a vertex cover of size } k\}$. Show that $VERTEX - COVER \in NP$.

Solution proposal Certificate: a subset of the nodes in G forming a vertex cover.

Exercise 7 Let $HAMCYCLE = \{\langle G\rangle \mid G \text{ is an undirected graph containing a Hamiltonian cycle}\}$.

Show that $HAMCYCLE \in NP$.

Solution proposal Certificate: a list of nodes from G forming a hamiltonian cycle.