INF 5300 Repetition

27.05.15
Anne Solberg

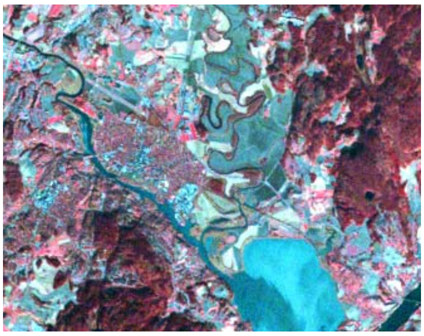
Energy functions for segmentation/classification

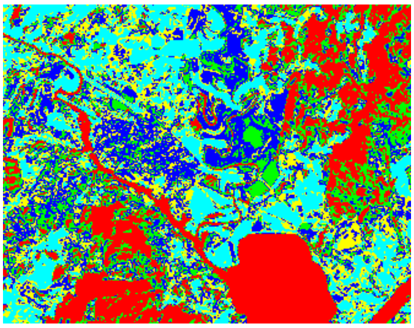
Anne Schistad Solberg

- Bayesian spatial models for classification
- Markov random field models for spatial context
- Other segmentation techniques:
  - EM-clustering
  - Mean shift segmentation
  - Graph-based segmentation (briefly)

Background – contextual classification

- An image normally contains areas of similar class
  - neighboring pixels tend to be similar.
- Classified images based on a non-contextual model often contain isolated misclassified pixels (or small regions).
- How can we get rid of this?
  - Majority filtering in a local neighborhood
  - Remove small regions by region area
  - Bayesian models for the joint distribution of pixel labels in a neighborhood.
- How do we know if the small regions are correct or not?
  - Look at the data, integrate spatial models in the classifier.





A Bayesian model for ALL pixels in the image

$Y = \{y_1,...,y_N\}$ Image of feature vectors to classify

$X = \{x_1,...x_N\}$ Class labels of pixels

- Classification consists choosing the class that maximizes the posterior probabilities for **ALL** pixels in the image

$$P(X \mid Y) = \frac{P(Y \mid X)P(X)}{\sum_{\text{all classes}} \mathrm{P}(\mathrm{Y} \mid \mathrm{X})\mathrm{P}(\mathrm{X})}$$

- Maximizing P(X|Y) with respect to $x_1,.....x_N$ is equivalent to maximizing P(Y|X)P(X) since the denominator does not depend on the classes $x_1,.....x_N$.
- Note: we are now maximizing the class labels of ALL the pixels in the image simultaneously.
- This is a problem involving finding N class labels simuntaneously.
- P(X) is the prior model for the scene. It can be simple prior probabilities, or a model for the spatial relation between class labels in the scene.

Back to the initial model...

Y = $\{y_1,\ldots,y_N\}$ Image of feature vectors to classify

X = $\{x_1,\ldots x_N\}$ Class labels of pixels

Task: find the optimal estimate **x**' of the true labels **x*** for all pixels in the image

- Classification consists choosing the class labels **x**' that maximizes the posterior probabilities

$$P(\mathbf{X}=\mathbf{x}\mid\mathbf{Y}=\mathbf{y})=\frac{P(\mathbf{Y}=\mathbf{y}\mid\mathbf{X}=\mathbf{x})P(\mathbf{X}=\mathbf{x})}{\sum_{\text{all classes}}\mathrm{P}(\mathbf{Y}=\mathbf{y}\mid\mathbf{X}=\mathbf{x})\mathrm{P}(\mathbf{X}=\mathbf{x})}$$

- We assume that the observed random variables are conditionally independent:

$$P(\mathbf{Y}=\mathbf{y}\mid\mathbf{X}=\mathbf{x})=\prod_{i=1}^{M}P(Y_i=y_i\mid X_i=x_i)$$

- We use a Markov field to model the spatial interaction between the classes (the term P(**X**=**x**)).

$$P(\mathbf{X}=\mathbf{x})=\mathrm{e}^{-\mathrm{U}(\mathbf{x})/\mathrm{Z}}$$

$$U(\mathbf{x})=\sum_{c\in Q}V_c(\mathbf{x})$$

$$V_c(\mathbf{x})=\beta I(x_i,x_k)$$

- Rewrite $P(Y_i=y_i|X_i=x_i)$ as

$$P(\mathbf{Y} = \mathbf{y} \mid \mathbf{X} = \mathbf{x}) = \frac{1}{Z_1} e^{-Udata(Y|X)}$$

$$Udata(Y \mid X) = \sum_{i=1}^{M} -\log P(Y_i = y_i \mid X_i = x_i)$$

- Then, $$P(\mathbf{X} = \mathbf{x} \mid \mathbf{Y} = \mathbf{y}) = \frac{1}{Z_2} e^{-Udata(Y|X)} e^{-U(X)}$$

- Maximizing this is equivalent to minimizing

$$U_{data}(Y \mid X) + U(X)$$

Udata(X|C)

- Any kind of probability-based classifier can be used, for example a Gaussian classifier with a k classes, d-dimensional feature vector, mean μ_k and covariance matrix Σ_k:

$$Udata(x_i \mid c_i) = -\frac{d}{2}\log(2\pi) - \frac{1}{2}\log(|\Sigma_k|) - \frac{1}{2} x_i^T \Sigma_k^{-1} x_i + \mu_k^T \Sigma_k^{-1} x_i - \frac{1}{2}\mu_k^T \Sigma_k^{-1} \mu_k$$

$$\propto -\frac{1}{2} x_i^T \Sigma_k^{-1} x_i + \mu_k^T \Sigma_k^{-1} x_i - \frac{1}{2}\mu_k^T \Sigma_k^{-1} \mu_k - \frac{1}{2}\log(|\Sigma_k|)$$

Finding the labels of ALL pixels in the image

- We still have to find an algorithm to find an estimate **x'** for all pixels.
- Alternative optimization algorithms are:
  - Simulated annealing (SA)
    - Can find a global optimum
    - Is very computationally heavy
  - Iterated Conditional Modes (ICM)
    - A computationally attractive alternative
    - Is only an approximation to the MAP estimate
  - Maximizing the Posterior Marginals (MPM)
- We will only study the ICM algorithm, which converges only to a local minima and is theoretically suboptimal, but computationally feasible.

ICM in detail

```
Initialize x_t, t=1,...N as the non-contextual classification by finding the class which maximize
      P(Y_t=y_t|X_t=x_t), assign it to classified_image(i,j)
For iteration k=1:maxit do
      For i=i:N,j=1:N (all pixels) do
            minimum_energy=High_number;
            For class s=1:S do
                  Udata = -log (P(Y_t=y_t|X_t=x_t))
                  Ucontxt=0;
                  nof_similar_neighbors=0;
                  for neighb=1:nof_neighbors
                        if (classified_image(neighb)=s) //neighbor and s of same class
                              ++nof_similar_neighbors;
                  Ucontxt = -beta*nof_similar_neighbors;
                  energy = Udata + Ucontxt;
                  if (energy < minimum_energy)
                              minimum_energy = energy;
                              bestclass = s;
            new_classified_image(i,j) = bestclass;
            if (new_classified_image(i,j)!=classified_image(i,j))
                  ++nof_pixels_changed;
      if nof_pixels_changed<min-limit
            break;
```

Clustering by mixtures of Gaussians

- Euclidean distance can be replaced by Mahalanobis distance from point x_i to cluster center k:

$$d\left(x_i, \mu_k, \Sigma_k\right) = \left(x_i - \mu_k\right)^T \Sigma_k^{-1} \left(x_i - \mu_k\right)$$

- We could just modify the K-means algorithm to use this measure after the first iteration.
- Mixtures of Gaussian considers that samples can be softly assigned to several nearby cluster centers:

$$p(x \mid \pi_k, \mu_k, \Sigma_k) = \sum_k \pi_k \frac{1}{|\Sigma_k|} e^{-d(x, \mu_k, \Sigma_k)}$$

- π_k is the mixing coefficient for cluster with mean μ_k and covariance Σ_k.

The EM-algoritm for clustering

- The EM-algoritm iteratively estimate the mixture parameters:

1. Expectation step (E-step): compute

$$z_{ik} = \frac{1}{Z_i} \pi_k \frac{1}{|\Sigma_k|} e^{-d(x, \mu_k, \Sigma_k)} \text{ with } \sum_k z_{ik} = 1$$

An estimate of the probability that xi belongs to the kth Gaussian

2. Maximation stage (M-step): update

$$\mu_k = \frac{1}{N_k} \sum_i z_{ik} x_i$$

$$\Sigma_k = \frac{1}{N_k} \sum_i z_{ik} (x_i - \mu_k)(x_i - \mu_k)^T$$

$$\pi_k = \frac{N_k}{N}$$

Mean shift clustering/segmentation algorithm

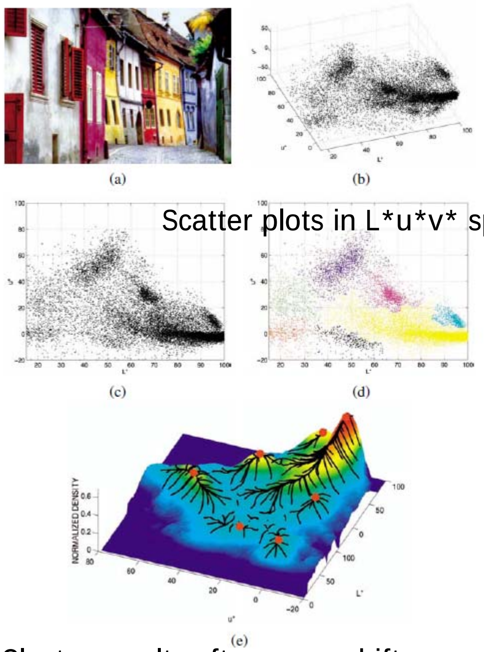
- K-means and mixtures of Gaussian are based on a parametric probability function.
- An alternative is to use a non-parametric smooth function that fits the data.
- The mean shift algoritms efficiently finds peaks in a distribution without estimating the entire distribution.
- It can be seen as the «inverse» of the watershed algorithm, which clims downhill.

The mean shift - background

- To estimate a density function for the scatter plots, we could use a Parzen window estimator, which smooths the data by convolving it with a kernel k() of width h:

$$f(x) = \sum_i K(x - x_i) = \sum_i k\left(\frac{\|x - x_i\|^2}{h^2}\right)$$

- When we have computed f(x), we could find peaks by gradient descent.
- Drawback: does not work well with sparse data points.
- Solution: finding the peaks WITHOUT estimating the entire distribution.



Cluster results after mean shift clustering, peaks marked in red

Mean shift segmentation

- Multiple restart gradient descent algorithm: start at many points y_k and take a step up-hill from these point.
- The gradient of f is (g(r)=-k'(r)):

$$\nabla f(x) = \sum_i (x_i - x) G(x - x_i) = \sum_i (x_i - x) g\left(\frac{\|x - x_i\|^2}{h^2} \right)$$

- This can be written as

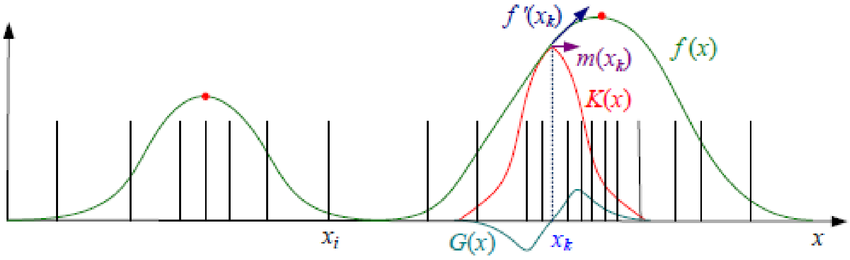
$$\nabla f(x) = \left[\sum_i G(x - x_i) \right] m(x)$$

$$m(x) = \frac{\sum_i x_i G(x - x_i)}{\sum_i G(x - xi)} - x$$

- The current estimate of y_k is replaced with its locally weighted mean:

$$y_{k+1} = y_k + m(y_k) = \frac{\sum_i x_i G(y_k - x_i)}{\sum_i G(y_k - x_i)}$$

Illustration of mean shift



- The kernel K is convolved with the image.
- The derivative of the kernel is computed by convolving the image with the derivative of the kernel
- The mean shift change m(x) is found from the derivative f'(x)

- Simple but slow algorithm: start a separate mean shift estimate y at every input point x, and iteration until only small changes.
- Faster: start at random points.
- Including location information:
  - Add the coordiates x_s= (x,y)in the kernel:

$$K(x_j) = k\left(\frac{\|x_r\|^2}{h_r^2}\right)k\left(\frac{\|x_s\|^2}{h_s^2}\right)$$

  - x_r is the spectral feature vector and h_r and h_s the bandwidth in the spectral and spatial domain.
  - The effect of this is that the algoritm step will take both spectral and spatial information and e.g. use larger steps in space between pixels with similar color.

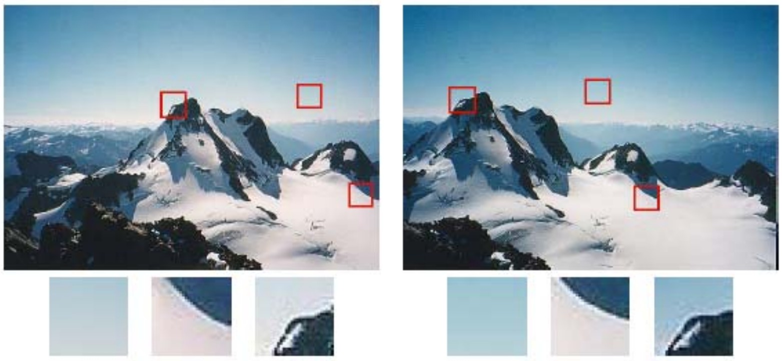
Detecting good features for tracking

Anne Schistad Solberg

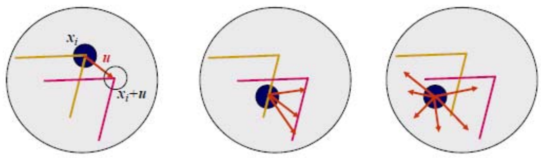
- Finding the correspondence between two images
  - **What are good features to match?**
    - Points?
    - Edges?
    - Lines?

Feature detection

- Goal: search the image for locations that are likely to be easy to match in a different image.



- What characterizes the regions? How unique is a location?
  - Texture?
  - Homogeneity?
  - Contrast?
  - Variance?



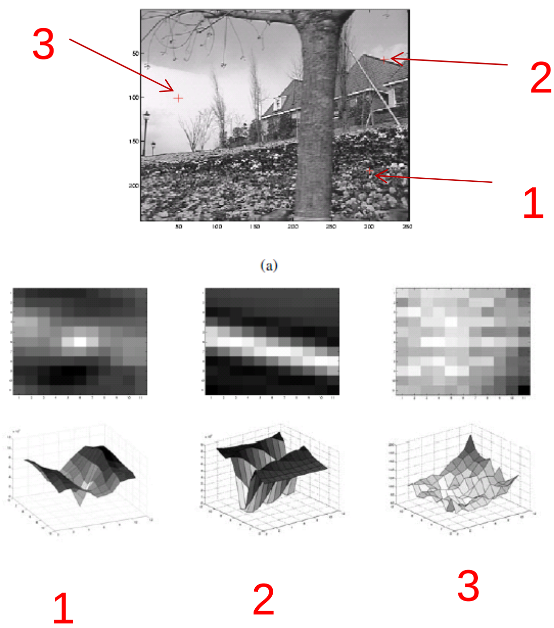
Feature detection

- A simple matching criterion: summed squared difference:

$$E_{WSSD}(u) = \sum_i w(x_i)\left[I_1(x_i + u) - I_0(x_i)\right]^2$$

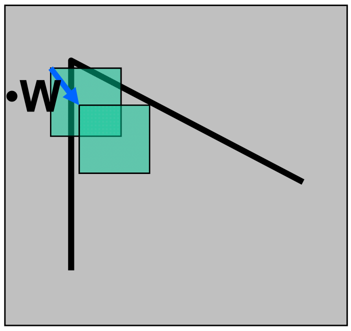
- I_0 and I_1 are the two images, **u**=(u,v) the displacement vector, and w(x) a spatially varying weight function.
- Check how stable a given location is (with a position change Δu) in the first image by computing the auto-correlation function:

$$E_{AC}(\Delta u) = \sum_i w(x_i)\left[I_0(x_i + \Delta u) - I_0(x_i)\right]^2$$



Feature detection: the math

- Consider shifting the window **W** by (u,v)
  - how do the pixels in **W** change?
  - Do a Taylor series expansion of the autocorrelation to allow fast computation:



$$
\begin{aligned}
E_{AC}(\Delta u) &= \sum_i w(x_i)[I_0(x_i+\Delta u) - I_0(x_i)]^2 \\
&\approx \sum_i \mathrm{w}(\mathrm{x_i})[\mathrm{I_0}(\mathrm{x_i}) + \nabla \mathrm{I_0}(\mathrm{x_i})\Delta u - I_0(x_i)]^2 \\
&= \sum_i \mathrm{w}(\mathrm{x_i})[\nabla \mathrm{I_0}(\mathrm{x_i})\Delta u]^2 \\
&= \Delta u^T \mathbf{A} \Delta u,
\end{aligned}
$$

where $\nabla \mathrm{I_0}(\mathrm{x_i}) = \left(\frac{\partial \mathrm{I_0}}{\partial \mathrm{x}}, \frac{\partial \mathrm{I_0}}{\partial \mathrm{y}}\right)(\mathrm{x_i})$ is the image gradient at $\mathrm{x_i}$.

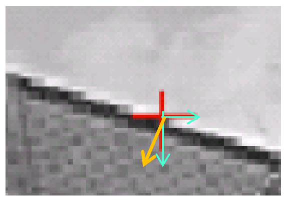
•Compute the gradients robustly using a Derivative of Gaussian filter

- The autocorrelation matrix A is:

$$
A = w * \begin{bmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{bmatrix}
$$

Feature detection: the math

- The matrix A carries information about the uncertainty of the location of a patch.
- A is called a tensor matrix and is formed by outer products of the gradients, convolved with a weighting function w to get a pixel-based uncertainty estimate.
- Eigenvector decomposition of A gives two eigenvalues, λ_0 and λ_1.
- The smallest eigenvalue carries information about the uncertainty.



•High gradient in the direction of maximal change

• If there is one dominant direction, we are quite certain about the direction estimate, and λ_{min} will be much smaller than λ_{max}.

•A high value of λ_{min} means that the gradient changes much in both directions, so this can be a good keypoint.

Feature detection: Harris corner detector

- Harris and Stephens (1988) proposed an alternative criterion computed from A (α=0.06 is often used):

$$\det(A) - \alpha\, \mathrm{trace}(A)^2 = \lambda_{\max}\lambda_{\min} - \alpha(\lambda_{\max} + \lambda_{\min})^2$$

- Other alternatives are e.g. the harmonic mean:

$$\frac{\det A}{\mathrm{trace}(A)} = \frac{\lambda_{\max}\lambda_{\min}}{\lambda_{\max} + \lambda_{\min}}$$

- The difference between these criteria is how the eigenvalues are blended together.

Feature detection algorithm

1. Compute the gradients I_x and I_y , and I_{xy} using a robust Derivative-of-Gaussian kernel (hint: convolve a Sobel x and y with a Gaussian).
2. Convolve these gradient images with a larger Gaussian to further robustify.
3. Create the matrix A from the robustified gradients from 2.
4. Compute either the smallest eigenvalue or the Harris corner detector measure from A.
5. Find local maxima above a certain threshold and report them as detected feature point locations.
6. Adaptive non-maximal suppression (ANMS) is often used to improve the distribution of feature points across the image.

How do we get rotation invariance?

- Option 1: use rotation-invariant feature descriptors.
- Option 2: estimate the locally dominant orientation and create a rotated patch to compute features from.

How do we get scale invariance?

- These operators look at a fine scale, but we might need to match features at a broader scale.
- Solution 1:
  - Create a image pyramid and compute features at each level in the pyramid.
    - At which level in the pyramid should we do the matching on? Different scales might have different characteristic features.
- Solution 2:
  - Extract features that are stable both in location AND scale.
  - SIFT features (Lowe 2004) is the most popular approach of such features.

Scale-invariant features (SIFT)

- See Distinctive Image Features from Scale-Invariant Keypoints by D. Lowe, International Journal of Computer Vision, 20,2,pp.91-110, 2004.
- Invariant to scale and rotation, and robust to many affine transforms.
- Main components:
  1. Scale-space extrema detection – search over all scales and locations.
  2. Keypoint localization – including determining the best scale.
  3. Orientation assignment – find dominant directions.
  4. Keypoint descriptor - local image gradients at the selected scale, transformed relative to local orientation.

SIFT: 1. Scale-space extrema

- The scale space is defined as the function $L(x,y,\sigma)$
- The input image is $I(x,y)$
- A Gaussian filter is applied at different scales $L(x,y,\sigma) = G(x,y,\sigma) * I(x,y,\sigma)$. σ is the scale.
- The Gaussian filter is:

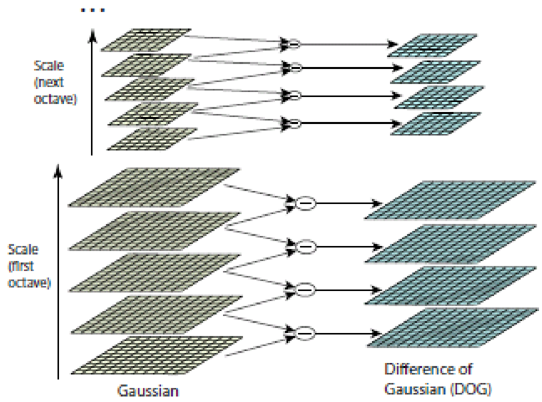
$$G(x,y,\sigma) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}$$

- Compute keypoints in scale space by difference-of-Gaussian, where the difference is between two nearby scales separated by a constant k:

$$D(x,y,\sigma) = (G(x,y,k\sigma) - G(x,y,\sigma)) * I(x,y) \\ = L(x,y,k\sigma) - L(x,y,\sigma)$$

- This is an efficient approximation of a Laplacian of Gaussian, normalized to scale σ. Lowe (2004) uses σ=1.6.

SIFT: 1. Scale-space extrema illustration

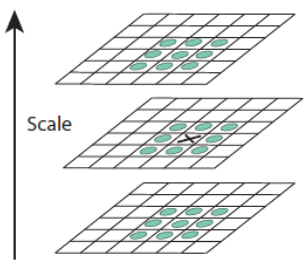


- For each octave of scale:
  - Convolve the image with Gaussians of different scale.
  - Compute Difference of Gaussians for adjacent Gaussians on a given octave.
- The next octave is down-sampled by a factor of 2.
- Each octave is divided into an integer number of scales s, $k=2^{1/s}$.

  This gives s+3 images in each octave.

SIFT 2 : accurate extrema detection

- First step in minimum/maximum detection: compare the value of $D(x,y,\sigma)$ to its 26 neighbors in this scale, and the scale above and below.c



- The candidate locations after this procedure are then checked for fit according to location, scale, and principal curvature.
- This is explained on the next slide.

SIFT 2: extrema detection

- Consider a Taylor series expansion of the scale-space function $D(x,y,\sigma)$ around sample point x

$$D(x) = D + \frac{\partial D^T}{\partial x} x + \frac{1}{2} x^T \frac{\partial^2 D}{\partial x^2} x$$

- The location of the extreme point is found by take the derivative of D(x) and setting it to zero:

$$\hat{x} = -\frac{\partial^2 D^{-1}}{\partial x^2} \frac{\partial D}{\partial x}$$

- It is computed by differences of neighboring sample points, yielding a 3x3 linear system.
- The value of D at the extreme point is useful for suppressing extrema with low contrast, $|D|<0.03$ are suppressed.

$$D(\hat{x}) = D + \frac{1}{2} \frac{\partial D^T}{\partial x} \hat{x}$$

SIFT 2: eliminating edge response based on curvature

- Since points on an edge are not very stable, such points need to be eliminated.
- This is done using the curvature, computed from the Hessian matrix of D.

$$H = \begin{bmatrix} D_{xx} & D_{xy} \\ D_{xy} & D_{yy} \end{bmatrix}$$

- The eigenvalues of H are proportation to principal curvatures of D. Consider the ratio between the eigenvalues α and β.
A good criteria is to only keep the points where

$$\frac{Tr(H)^2}{Det(H)} < \frac{(r+1)^2}{r}$$

- r=10 is often used.

SIFT 3: computing orientation

- To normalize for the orientation of the keypoints, we need to estimate the orientation. The feature descriptors (next step) will then be computed relative to this orientation.
- They used the gradient magnitude m(x,y) and direction θ(x,y) to do this (L is a Gaussian smoothed image at the scale where the keypoints were found).

$$m(x,y)=\sqrt{(L(x+1,y)-L(x-1,y))^2+(L(x,y+1)-L(x,y-1))^2}$$
$$\theta(x,y)=\tan^{-1}\left(\left(L(x,y+1)-L(x,y-1)\right)/\left(L(x+1,y)-L(x-1,y)\right)\right)$$

- Then, they computed histograms of the gradient direction, weighted by gradient magnitude. The histograms are formed from points in the neighborhood of a keypoint.
- 36 bins covers the 360 degrees of possible orientations.
- In this histogram, the highest peak, and other peaks with height 80% of max are found. If a localization has multiple peaks, it can have more than 1 orientation.
- WHY are locations with more than one orientation important?

Feature descriptors

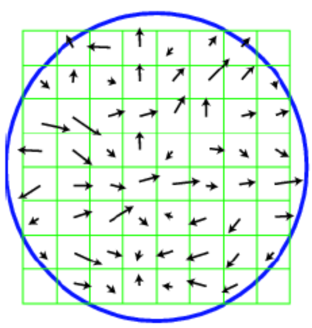
- Which features should we extract from the key points?
- These features will later be used for **matching** to establish the motion between two images.
- How is a good match computed (more in chapter 8)?
  - Sum of squared differences in a region?
  - Correlation?
- The local appearance of a feature will often change in orientation and scale (this should be utilized e.g. by extracting the local scale and orientation and then use this scale (or a coarser one) in the matching).

SIFT 4: feature extraction stage

- Given
  - Keypoint locations
  - Scale
  - Orientation for each keypoint
- What type of features should be used for recognition/matching?
  - Intensity features? Use correlation as match?
  - Gradient features?
    - Similar to our visual system. SIFT uses gradient features.

SIFT: feature extraction stage

- Select the level of the Gaussian pyramid where the keypoints were identified.
- Main idea: use histograms of gradient direction computed in a neighborhood as features.
- Compute the gradient magnitude and direction at each point in a 16x16 window around each keypoint. Gradients should be rotated relative to the assigned orientation.
- Weight the gradient magnitude by a Gaussian function.

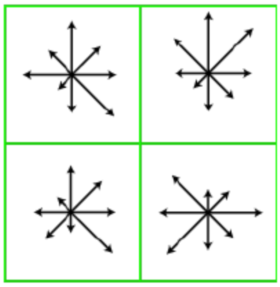


(a) image gradients

Each vector represents the gradient magnitude and direction. The circle illustrates the Gaussian window.

SIFT 4: feature extraction stage

- Form a gradient orientation histogram for each 4x4 quadrant using 8 directional bins. The value in each bin is the sum of the gradient magnitudes in the 4x4 window.
- Use trilinear interpolation of the gradient magnitude to distribute the gradient information into neighboring cells.
- This results in 128 (4x4*8) non-negative values which are the raw SIFT-features.
- Further normalize the vector for illumination changes and threshold extreme values.



(b) keypoint descriptor

This illustration shows a 2x2 descriptor array and not 4x4

Feature matching

- Matching is divided into:
  - Define a matching strategy to compute the correspondence between two images.
  - Using efficient algorithms and data structures for fast matching (we will not go into details on this).
- Matching can be used in different settings:
  - Compute the correspondende between two partly overlapping images (= stitching).
    - Most key points are likely to find a match in the two images.
  - Match an object from a training data set with an unknown scene (e.g. for object detection).
    - Finding a match might be unlikely

Computing the match

- Assume that the features are normalize so we can measure distances using Euclidean distance.
- We have a list of keypoints features from the two images. Given a keypoint in image A, compute the similarity (=distance) between this point and all keypoints in image B.
- Set a threshold to the maximum allowed distance and compute matches according to this.
- Quantify the accuracy of matching in terms of:
  - TP: true positive: number of correct matches
  - FN: false negative: matches that were not correctly detected.
  - FP: false positive: proposed matches that are incorrect.
  - TN: true negative: non-matches that were correctly rejected.

Feature-based alignment

Anne Schistad Solberg

- Finding the alignment between features from different images
- Geometrical transforms – short repetition
- RANSAC algorithm for robust transform computation

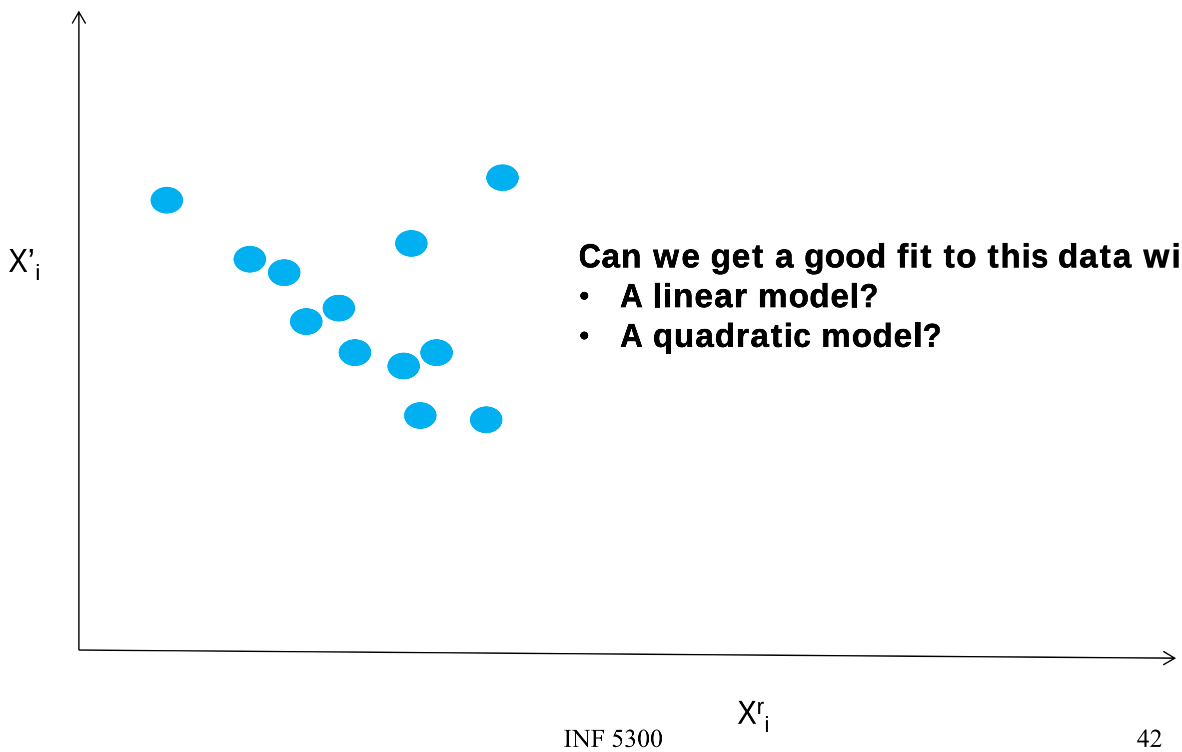
INF 2310 - coregistration III

- The root mean square error is used to evaluate how good a match is
- Given M point pairs $(x_i,y_i),(x_i^r,y_i^r)$ (r is the reference image)
- Assume that the transform gives estimated coordinates in the reference image as (x'_i,y'_i)
- (x_i,y_i) --> (x'_i,y'_i)
- The number of point pairs is M >>3 for affine transforms og M>>6 for quadratic
- The coefficients in the transform are computed as the values that minimize the square error between the true coordinates
- (x_i^r,y_i^r) and the transformed coordinates (x_i',y_i')

$$J = \sum_{i=1}^{M} (x_i' - x_i^r)^2 + (y_i' - y_i^r)^2$$

- Simple linear algebra is used to find the solution to this problem.

A data example
Estimated vs. true coordinates



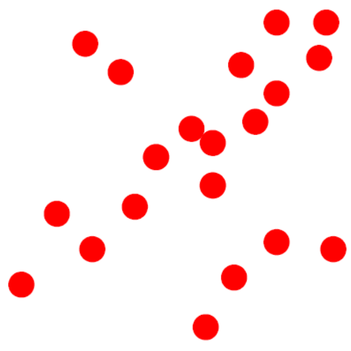
Can we get a good fit to this data with
- **A linear model?**
- **A quadratic model?**

Introducing a robust matching algorithm

- The detected features are not perfect, there may be outliers where the match is NOT good.
- If we want to fit a line:
  - Count the number of points that agree with the line.
    - Agree means that the distance between the location of the estimated and the true coordinates is very small.
    - Points which fulfill this criterion are called inliers.
    - Other points are called outliers.
  - For all possible lines, select the one with the larges number of inliers.

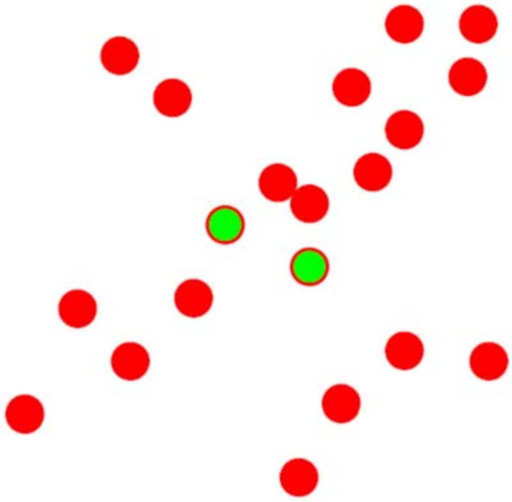
RANSAC

- RANdom Sample Consensus (Fischler and Bolles, 1981)
- Algorithm:
  1. Sample (randomly) exactly the number of points needed to fit the model.
  2. Solve for the model parameters based on the samples.
  3. Score by the fraction of inliers within a preset threshold.
- Repeat 1-3 until the best model is found with high confidence.



RANSAC

Line fitting example



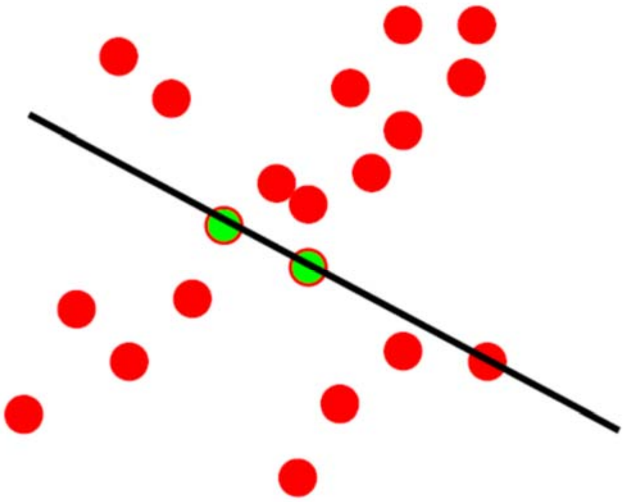
Algorithm:

1. **Sample** (randomly) the number of points required to fit the model (#=2)
2. **Solve** for model parameters using samples
3. **Score** by the fraction of inliers within a preset threshold of the model

Repeat 1-3 until the best model is found with high confidence

RANSAC

Line fitting example



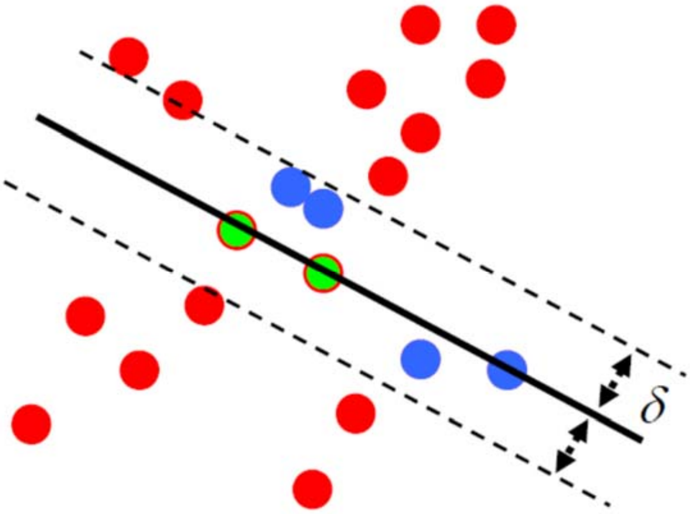
Algorithm:

1. **Sample** (randomly) the number of points required to fit the model (#=2)
2. **Solve** for model parameters using samples
3. **Score** by the fraction of inliers within a preset threshold of the model

Repeat 1-3 until the best model is found with high confidence

RANSAC

Line fitting example



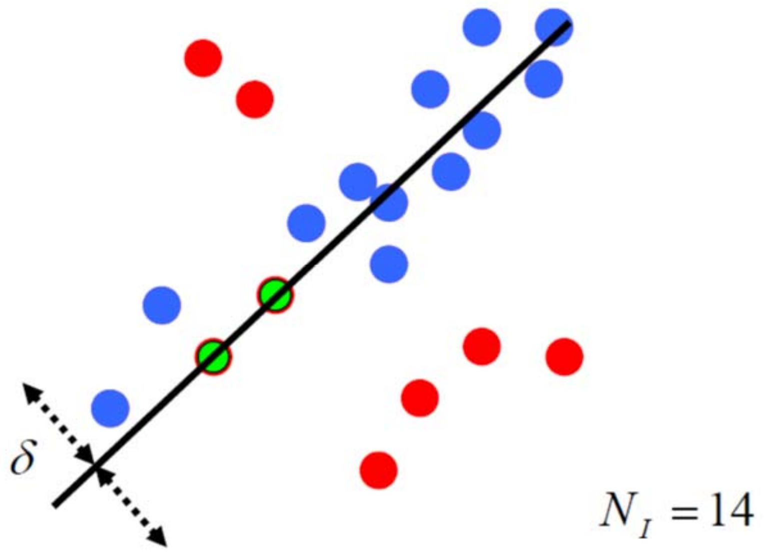
$$N_I = 6$$

Algorithm:

1. **Sample** (randomly) the number of points required to fit the model (#=2)
2. **Solve** for model parameters using samples
3. **Score** by the fraction of inliers within a preset threshold of the model

Repeat 1-3 until the best model is found with high confidence

RANSAC



$$N_I = 14$$

Algorithm:

1. **Sample** (randomly) the number of points required to fit the model (#=2)
2. **Solve** for model parameters using samples
3. **Score** by the fraction of inliers within a preset threshold of the model

Repeat 1-3 until the best model is found with high confidence

RANSAC algorithm

General version:

1. Randomly choose s samples
   s=minimum sample size that let you fit a model
2. Fit a model (e.g. line) to those samples
3. Count the number of inliers that approximately fit the model.
4. Repeat N times
5. Choose the model that has the largest set of inliers, and fit this model to all inliers using e.g. least squares.
   – When we have the best set of points, refine the model using all inliers.

RANSAC conclusions

- Good:
  – Robust to outliers (can handle up to 50% outliers)
  – Applicapable to a larger number of parameters than Hough transform/parameters are easier to choose.
- Bad:
  – Computational time grows quickly with fraction of outliers and number of parameters.
  – Not good for getting multiple fits.
- Common applications:
  – Robust linear regression (and similar)
  – Computing the transform behind image stitching (called homography)
  – Image registration/Estimating the fundamental matrix relating two views.

Dense motion and flow

Anne Schistad Solberg

- Motion perception
- Motion visualization
- Image similarity measures
- Motion estimation
- Optical flow algorithm

Essential steps in motion estimation

- An error metric to compare the two images must be chosen.
- A search technique to compute the best match is needed.
  - Pyramid search is often used to speed up the process.
- Accurate motion estimates might need subpixel accuracy.

- Regularization is often applied since the motion vectors are not reliable in all regions.
  - For compex motion layered motion models might also be needed.

Matching criteria

- What is invariant between the two images?
  - Brightness? Gradients? Phase? Other features?
- Distance metric: (L2,L1, truncated L1, Lorentzian)

$$E(u,v) = \sum_{x,y} \rho\left(I_1(x,y) - I_2(x+u,y+v)\right)$$

- Correlation, normalized cross correlation

Computing similarity between image patches

- A simple matching criterion: summed squared difference (**SSD**):

$$E_{SSD}(u) = \sum_i [I_1(x_i + u) - I_0(x_i)]^2$$

- I_0 and I_1 are the two images, **u**=(u,v) the displacement vector.
- Movement can be at the sub-pixel level so interpolation might be needed.
- A measure more robust to outliers is

$$E_{SRD} = \sum_i \rho\left(I_1(x_i + u) + I_0(x_i)\right)$$

$$\rho(x) = \frac{x^2}{1 + \frac{x^2}{a^2}}$$

  - a is a constant called outlier threshold

Hierarchical search for matches – block matching

- In motion estimation, there are often small motion between frames, so the search is restricted to a small region (e.g. ±16 pixels) from a given position.
- This is called block matching.
- Hiearchical motion estimation is often used to speed up the process.
  - An image pyramid is created by decimation and smoothing, consisting of images.

$$I_k^{(l)}\left(x_j\right) \leftarrow \tilde{I}_k^{(l-1)}\left(2x_j\right)$$

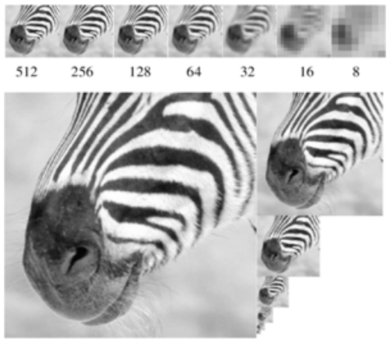
$\tilde{I}_k^{(l-1)}\left(2x_j\right)$ is a smoothed version of the image at level l-1

  - At the coarsest level, we do a full search in a window for the displacement u(l) that minimizes

$$I_o^{(l)} - I_1^{(l)}$$

  - This value of the motion vector is then used to predict the displacement at the finer level:

$$\hat{u}^{(l-1)} \leftarrow 2u^{(l)}$$



Lucas and Kanade optical flow

- Good image stabilization requires subpixel accuracy.
- Assume that matching is based on the SSD-criterion.
- Lucas and Kanade did a gradient descent on the SSD function to refine the shift in u based on a Taylor series expansion of $I_1(x_i+u+\Delta u)$.
- Let

$$J_1(x_i+u) = \nabla I_1(x_i+u)$$

- Define

$$e_i = I_1(x_i+u) - I_0(x_i)$$

- The modified SSD criterion is then:

$$E_{LK-SSD}(u+\Delta u) = \sum_i \left[I_1(x_i+u+\Delta u) - I_0(x_i)\right]^2$$

$$\approx \sum_i \left[I_1(x_i+u) + J_1(x_i+u)\Delta u - I_0(x_i)\right]^2$$

$$= \sum_i \left[J_1(x_i+u)\Delta u + e_i\right]^2$$

Gradient Constraint (or the Optical Flow Constraint)

$$E(u,v) = (I_x \cdot u + I_y \cdot v + I_t)^2$$

Minimizing:

$$\frac{\partial E}{du} = \frac{\partial E}{dv} = 0$$

$$I_x(I_x u + I_y v + I_t) = 0$$

$$I_y(I_x u + I_y v + I_t) = 0$$

In general $I_x, I_y \neq 0$

Hence, $I_x \cdot u + I_y \cdot v + I_t \approx 0$

Least-square problem, see Appendix A.2 for details

Weighted version

A weight function can be used to weight constraints in the center of the neighborhood with a gaussian function g(x)

$$E_W(u,v) = \sum_{x,y\in\Omega} g(x)\left(I_x(x,y)u + I_y(x,y)v + I_t\right)^2$$

Minimizing

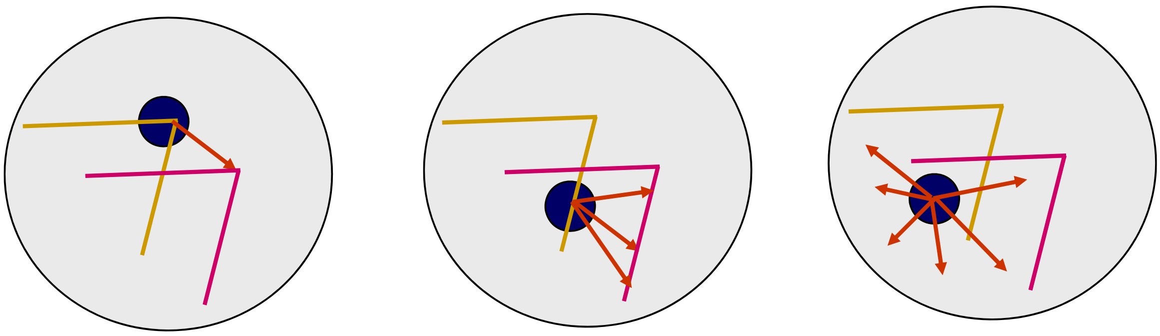
$$\begin{bmatrix} \sum gI_x^2 & \sum gI_xI_y \\ \sum gI_xI_y & \sum gI_y^2 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} \sum gI_xI_t \\ \sum gI_yI_t \end{pmatrix}$$

Balance spatial gradients by temporal gradients and the shift in u

Note: to compute the derivatives, take care so they center at the same location (e.g. I(x,y)-I(x+1,y) will not center at the same location in all directions.

Local Patch Analysis

- How *certain* are the motion estimates?
- This is similar to finding good keypoints in SIFT.



The Aperture Problem

Let $A = \sum (\nabla I)(\nabla I)^T$ and $b = \begin{bmatrix} -\sum I_x I_t \\ -\sum I_y I_t \end{bmatrix}$

- Algorithm: At each pixel compute U by solving $AU = b$
- *A* is singular if all gradient vectors point in the same direction
  - e.g., along an edge
  - of course, trivially singular if the summation is over a single pixel or there is no texture
  - i.e., only *normal flow* is available (aperture problem)
- Corners and textured areas are OK

Refining the search to sub-pixel accuracy

- Estimate velocity at each pixel using one iteration of Lucas and Kanade estimation.
- Many applications, like image stabilization and stitching, require sub-pixel accuracy in matching.
- Refine this estimate by repeating the process
- Remember that the Taylor series expansion ignored the higher order terms
  - The accuracy of the estimate is bounded by the magnitude of the displacement and the second derivative of I.
- If we undo the motion, and reapply the estimator to the warped signal to find the residual motion left
  - Do this iteratively until the residual motion is small
  - Let ut now explain this

Limits of the gradient method

Fails when intensity structure in window is poor

Fails when the displacement is large (typical operating range is motion of 1 pixel)

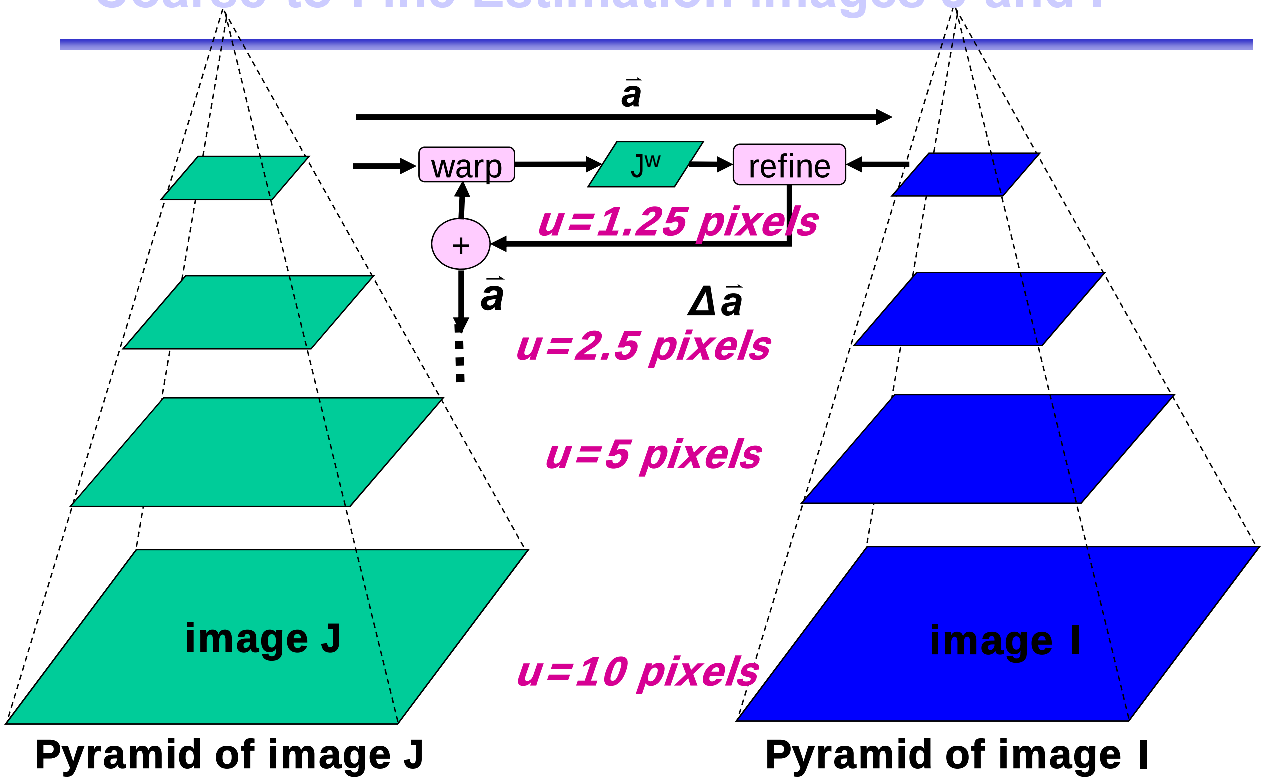
Linearization of brightness is suitable only for small displacements

- Also, brightness is not strictly constant in images

actually less problematic than it appears, since we can pre-filter images to make them look similar

Szeliski

Coarse-to-Fine Estimation images J and I



Parametric motion models (8.2)

- *2D Models:*
- Affine
- Quadratic
- Planar projective transform (Homography)

- *3D Models (see the book):*
- Instantaneous camera motion models
- Homography+epipole
- Plane+Parallax

Example: Affine Motion

$$u(x, y) = a_1 + a_2 x + a_3 y$$
$$v(x, y) = a_4 + a_5 x + a_6 y$$

- Substituting into the brightness constraint equation:

$$\boxed{I_x (a_1 + a_2 x + a_3 y) + I_y (a_4 + a_5 x + a_6 y) + I_t \approx 0}$$

Each pixel provides 1 linear constraint in 6 *global* unknowns

Least Square Minimization (over all pixels):

$$\boxed{Err(\vec{a}) = \sum \left[I_x (a_1 + a_2 x + a_3 y) + I_y (a_4 + a_5 x + a_6 y) + I_t \right]^2}$$

Learning goals – motion estimation

- Understand representation and visualization of motion vectors.
- Understand the brightness similarity criterion.
- Know different patch similarity measures.
- Understand the gradient constraint.
- Know the basic steps in the optical flow algorithm
- Know strenghts and limitations of optical flow

Snakes

The energy function

$$E_{snake} = \int_{s=0}^{1} E_{\text{int}}(v(s)) + E_{image}(v(s)) + E_{con}(v(s))ds$$

Internal deformation energy of the snake itself. How it can bend and stretch.

Constraints on the shape of the snake. Enchourages the contour to be smooth. (Often omitted)

A term that relates to gray levels in the image, e.g. attracts the snake to points with high gradient magnitude.

The minimum values is found by derivation:

$$\frac{dE_{snake}}{dv} = 0$$

The internal deformation term

$$E_{\text{int}} = \alpha(s)\left|\frac{dv(s)}{ds}\right|^2 + \beta(s)\left|\frac{d^2v(s)}{ds^2}\right|^2$$

First derivative
Measures how stretched the contour is.
Keyword: point spacing.
Imposes tension.
The curve should be short if possible.
Physical analogy: v acts like a membrane.

Second derivative
Measures the curvature or bending energy.
Keyword: point variation.
Imposes rigidity.
Changes in direction should be smooth.
Physical analogy: v acts like a thin plate.

α and β are penalty parameters that control the weight of the two terms.
Low α values: the snake can stretch much.
Low β values: the snake can have high curvature.

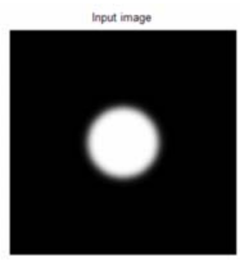
A simple image term

$$E_{image} = \int_0^1 P(v(s))ds$$

- A common way of defining P(x,y) is:

$$P(x,y) = -c\left|\nabla(G_\sigma * I(x,y))\right|$$

- c is a constant, ∇ is a gradient operator, G_σ is a Gaussian filter, and I(x,y) the input image. Note the minus sign as the gradient is high for edges.



The energy function

- Simple snake with only two terms (no termination energy):

$$E_{snake}(s) = E_{\text{int}}(v_s) + E_{image}(v_s)$$
$$= \alpha \left| \frac{dv_s}{ds} \right|^2 + \beta \left| \frac{d^2 v_s}{ds^2} \right|^2 + \gamma E_{edge}$$

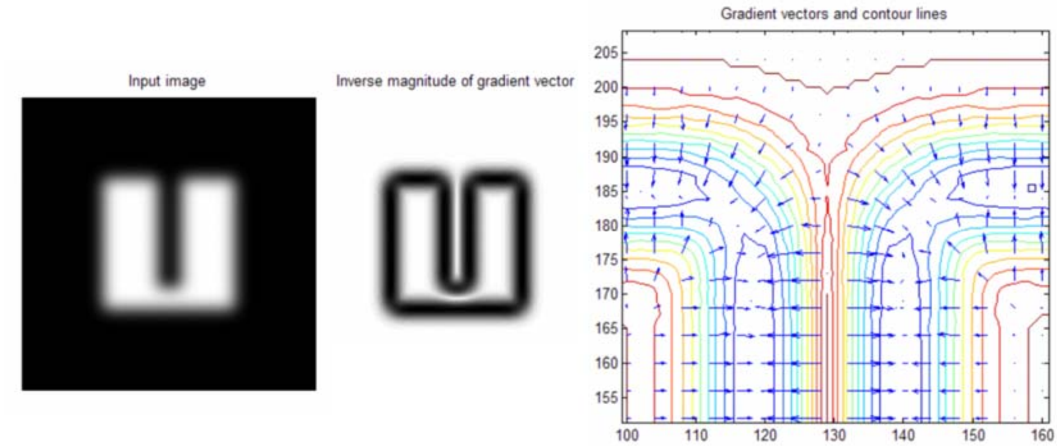
- We need to approximate both the first derivative and the second derivative of v_s, and specify how E_{edge} will be computed.
- How should the snake iterate from its initial position?

How do we implement this?

- The energy function involves finding the new location of S new coordinates (x_s,y_s), $0 \leq s \leq 1$ for one iteration.
- Which algorithm can we use to find the new coordinate locations?
  1. Greedy algorithm
     – Simple, suboptimal, easier to understand
  2. Complete Kass algorithm
     – Optimizes all points on the countour simultaneously by solving a set of differential equations.

– These two algorithms will now be presented.

Capture range problems



Capture range problems

$$\mu\nabla^2 u = 0$$
$$\mu\nabla^2 v = 0$$

- The first term is Lagrange's equation which appear in often in physics, e.g. in heat flow or fluid flow.
- Imaging the a set of heaters is initialized at certain boundary conditions. As time evolves, the heat will redistribute/diffuse until we reach an equilibrium.
- In our setting, the gradient term act as the starting conditions.
- As the differential equation iterate, the gradient will diffuse gradually to other parts of the image in a smooth manner.

Capture range problems

- This equation has a similar solution to the original differential equation.
- We treat u and v as functions of time and solve the equations iteratively.
  – Comparable to how we iteratively computed $x^{<i+1>}, y^{<i+1>}$ from $x^{<i>}, y^{<i>}$
- The solution is obviously a numerical one, we use two sets of iterations, one for u and one for v.
- After we have computed v(x,y), we replace E_{ext} (the edge magnitude term) by v(x,y)
- So an interative algorithm is first used to compute v(x,y)

Computing **v**(x,y) continued..

- Select a time step Δt and a pixel spacing Δx and Δy for the iterations.
- Approximate the partial derivatives as

$$u_t = \frac{1}{\Delta t}\left(u_{i,j}^{n+1} - u_{i,j}^{n}\right)$$

$$v_t = \frac{1}{\Delta t}\left(v_{i,j}^{n+1} - v_{i,j}^{n}\right)$$

$$\nabla^2 u = \frac{1}{\Delta x \Delta y}\left(u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j}\right) \text{A Laplacian approximation}$$

$$\nabla^2 v = \frac{1}{\Delta x \Delta y}\left(v_{i+1,j} + v_{i,j+1} + v_{i-1,j} + v_{i,j-1} - 4v_{i,j}\right)$$

- Then the iterative equations are:

$$u_{i,j}^{n+1} = \left(1 - b_{i,j}\Delta t\right)u_{i,j}^{n} + r\left(u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j}\right) + c_{i,j}^{1}\Delta t$$

$$v_{i,j}^{n+1} = \left(1 - b_{i,j}\Delta t\right)v_{i,j}^{n} + r\left(v_{i+1,j} + v_{i,j+1} + v_{i-1,j} + v_{i,j-1} - 4v_{i,j}\right) + c_{i,j}^{2}\Delta t$$

$$r = \frac{\mu \Delta t}{\Delta x \Delta y}$$

To get convergence we must have

$$\Delta \mathrm{t} \le \frac{\Delta x \Delta y}{4\mu}$$

Capture range problems