Wavenumber-Frequency Space

- Four-dimensional Fourier transform:
  \[ S(\overrightarrow{k}, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(x, t) \exp\{ j(\omega t - \overrightarrow{k} \cdot \overrightarrow{x}) \} d\overrightarrow{x} dt \]

- Spatial frequency variable \( \overrightarrow{k} \) dual to \( \overrightarrow{x} \)
- Just like temporal frequency, \( \omega \), and \( t \)
- Note the different signs for the exponents, due to the concern with propagating waves
Inverse Fourier transform

\[ s(\vec{x}, t) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(\vec{k}, \omega) \exp\{j(\omega t - \vec{k} \cdot \vec{x})\} d\vec{k} d\omega \]

- Any reasonable spatiotemporal signal can be decomposed into infinitely many plane waves as long as the Fourier transform converges.
- This also holds for an arbitrary, nonperiodic, nonpropagating distribution of energy in space and time. It can be represented as a superposition of periodic, propagating plane waves.

Monochromatic Plane Wave

\[ s(\vec{x}, t) = e^{j(\omega_0 t - \vec{k}^0 \cdot \vec{x})} \]

- Fourier transform:

\[ S(\vec{k}, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\{j(\omega t - \vec{k}^0 \cdot \vec{x})\} \exp\{-j(\omega t - \vec{k} \cdot \vec{x})\} d\vec{x} dt \]

- Separable, product of four factors:

\[ \int_{-\infty}^{\infty} \exp\{-j(\omega - \omega_0)t\} dt = \delta(\omega - \omega_0) \]

- Result: A single point in \((\vec{k}, \omega)\) space

\[ S(\vec{k}, \omega) = \delta(\vec{k} - \vec{k}^0) \cdot \delta(\omega - \omega_0) \]
Propagating Wave

\[ s(\vec{x}, t) = s(t - (\vec{k}^0 / \omega) \cdot \vec{x}) = s(t - \vec{\alpha}^0 \cdot \vec{x}) \]

- Only energy along a line in wavenumber-frequency space

\[ S(\vec{k}, \omega) = S(\omega)\delta(\vec{k} - \vec{k}^0) = S(\omega)\delta(\vec{k} - \omega \vec{\alpha}^0) \]

- This line is given by the dispersion equation for the wave in the medium

\[ \vec{k}^0 = \omega \vec{\alpha}^0 \]

- dispersionless medium \( |\vec{k}| = \omega / c \)

---

Wavenumber-Frequency Space

Figure 2.14: When \( S(\vec{k}, \omega) \) contains temporal frequencies near \( \omega_0 \) only (left portion), the wavenumber-frequency spectrum \( S(\vec{k}, \omega) \) has significant energy only on the plane \( \omega = \omega_0 \) in \((\vec{k}, \omega)\) space. Here we have displayed the three-dimensional space \((\vec{k}, \omega)\) rather than the full four-dimensional space \((\vec{k}_x, \vec{k}_y, \vec{k}_z, \omega)\) for the purposes of illustration. If a signal consists of components propagating in a particular direction \( \vec{\alpha} \) (right portion), then its wavenumber-frequency spectrum is zero except for the half-plane where \( \vec{k} \) is proportional to \( \vec{\alpha} \).
2.5.3 Spectrum – Spherical Wave

- Only of theoretical interest
- Skip it
Filtering in Wavenumber-Frequency Space

- Straightforward extension to spatiotemporal signals:
  \[ Y(\vec{k}, \omega) = H(\vec{k}, \omega) \cdot X(\vec{k}, \omega) \]

- 4-D convolution integral:
  \[ y(\vec{x}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\vec{x} - \vec{\xi}, t - \tau) s(\vec{\xi}, \tau) d\vec{\xi} d\tau \]

- Same considerations as with time-domain LTI-filters: Ideal filters \( \leftrightarrow \) infinitely long impulse responses

- Ideal filters:
  - Focus on one frequency \( H(\cdot) = \delta(\omega - \omega_0) \)
  - Or focus in one direction \( H(\cdot) = \delta(k - k_0) \)

- Beamforming = realizable space-time filtering
  - Linear filtering in chapter 4
  - Nonlinear methods in chapter 7

2.6 Random Fields

- Will go through it if needed in chapter 7