Aperture and Arrays

- Study apertures: Examine the effect of sensors that gather signal energy over finite areas.
- Arrays: Group of sensors combined to produce single output.
- At m'th sensor position, $\vec{x}_m$:
  - Fields value: $f(\vec{x}_m, t)$.
  - Sensors output: $y_m(t)$.
  - If sensor is *perfect* (i.e. linear transf., infinite bandwidth, omni-directional):
    $$y_m(t) = \kappa \cdot f(\vec{x}_m, t), \kappa \in \mathbb{R} \text{ (or } \mathbb{C}).$$

Aperture function and aperture smoothing function

- Directional ↔ omni-directional
  - If sensor has (significant) spatial extent, it will spatially integrate energy, i.e. it focus in a particular propagation direction.
  - Example: Parabolic dish.
- Apertures (≠ point sources) are described by the aperture function, $w(\vec{x})$.
  - Spatial extend reflects size and shape
  - Aperture weighting; relative weighting of the field within the aperture (also known as shading, tapering, apodization).
- Aperture smoothing function:
  $$W(\vec{k}) = \int_{-\infty}^{\infty} w(\vec{x}) \exp(j\vec{k} \cdot \vec{x}) d\vec{x}$$
Aperture smoothing function (≠ Sec. 3.1.1)

- Given
  - Aperture: \( w(\vec{x}) \)
  - Field: \( f(\vec{x}, t) \)
  - Linear sensor
  - Contribution from area \( \delta \vec{\xi} \) at \( \vec{\xi} \): \( w(\vec{\xi}) f(\vec{\xi}, t) \delta \vec{\xi} \)
  - Contribution from sensor \( z(t) = \int_{\text{aperture}} w(\vec{\xi}) f(\vec{\xi}, t) \delta \xi \).

Aperture smoothing function ...

- Assume aperture in \( \vec{x} \)

- \( z(\vec{x}, t) = \int_{\vec{\xi}} w(\vec{\xi} - \vec{x}) f(\vec{\xi}) \delta \xi = w(-\vec{x}) * f(\vec{x}, t) \),
  (i.e. spatial correlation)
  should have been spatial convolution
  \( \leftrightarrow \) (space-time F.T.)
  \( Z(\vec{k}, w) = \int_{\vec{x}} \int_{t} z(\vec{x}, t) e^{i\vec{k} \cdot \vec{x}} \ dt \ d\vec{x} \)
  \( = W(-\vec{k}) F(\vec{k}, w) \)
  Must use symmetry assumption ...
  \( = W(\vec{k}) F(\vec{k}, w) \)
  This differs from Eq. (3.1)!

Assume a single plane wave, propagating in direction \( \vec{\zeta_0} \), \( \vec{\zeta_0} = \vec{k_0} / k \)

\( f(\vec{x}, t) = s(t - \vec{a}^0 \cdot \vec{x}) \), \( \vec{a}^0 = \vec{\zeta_0} / c \)

\( F(\vec{k}, w) = S(w) \delta(\vec{k} - w\vec{a}^0) \) (Sec. 2.5.1)

This prop. wave contains energy only along the line \( \vec{k} = w\vec{a}^0 \) in wavenumber-frequency space.
Aperture smoothing function ...

- Linear aperture:
  \( b(x) = 1, \ |x| \leq D/2 \)
  \[ W(\vec{k}) = \frac{\sin k_x D/2}{k_x/2} \]
- Rectangular aperture:
  \( w(x, y) = b_1(x)b_2(y) \)
  \[ W(k_x, k_y) = W(x)W(y) \]
  \[ W(k_x, k_y) = \frac{\sin k_x D_x/2 \sin k_y D_y/2}{k_x/2 \ k_y/2} \]
- Circular aperture:
  \( o(x, y) = 1, \ \sqrt{x^2 + y^2} \leq R \)
  \[ O(k_{xy}) = \frac{2\pi R}{k_{xy}} J_1(k_{xy} R) \]

Characterizing \( W(\vec{k}) \)

- Linear aperture:
  1. sidelobe at \( k_{x0} \approx 2.86\pi/D \)
  \[ |W(k_{x0})| \approx 0.2172D \Rightarrow \]
  \[ ML_{SL} \approx \frac{D}{2172} = 4.603 \times 13.3\text{dB} \]
- Circular aperture:
  1. SL at \( k_{xy0} \approx 5.14/R \)
  \[ ML_{SL} \approx 7.56 \times -17.57\text{dB} \]
- Projection-slice theorem

Classical resolution

- Spatial extent of \( w(\vec{x}) \) determines the resolution with which two plane waves can be separated.
- Ideally, \( W(\vec{k}) = \delta(\vec{k}) \), i.e. infinite spatial extent!

Rayleigh criterion:

Two incoherent plane waves, propagating in two slightly different directions, are resolved if the mainlobe peak of one aperture smoothing function replica falls on the first zero of the other aperture smoothing function replica, i.e. half the mainlobe width.

Classical resolution ...

- Linear aperture of size \( D \)
  \[ W(k_x) = \frac{\sin(k_x D/2)}{k_x/2} (= D\text{sinc}(k_x D/2)) = \frac{\sin(\pi \sin \theta \lambda / \lambda)}{\pi \sin \theta / \lambda} \]
  - 3 dB width: \( \theta_{-3dB} \approx 0.89\lambda / D \)
  - 6 dB width: \( \theta_{-6dB} \approx 1.21\lambda / D \)
  - Zero-to-zero distance: \( \theta_{0-0} \approx 2\lambda / D \)
- Circular aperture of diameter \( D \)
  \[ W(k_{xy}) = \frac{2\pi D/2}{k_{xy}} J_1(k_{xy} D/2) \]
  - 3 dB width: \( \theta_{-3dB} \approx 1.02\lambda / D \)
  - 6 dB width: \( \theta_{-6dB} \approx 1.41\lambda / D \)
  - Zero-to-zero distance: \( \theta_{0-0} \approx 2.44\lambda / D \)
- Rule-of-thumb; Angular resolution: \( \theta = \lambda / D \)
Geometrical optics

- Validity: down to about a wavelength
- Near field-far field transition
  - $d_{90} = D^2/\lambda$ for a maximum phase error of $\lambda/8$ over aperture
- $f$-number
  - Ratio of range and aperture: $f_\# = R/D$
- Resolution
  - Angular resolution: $\theta = \lambda/D$
- Depth of focus
  - Aperture is focused at range $R$. Phase error of $\lambda/8$ yields $r = \pm f_\#^2 \lambda$ or DOF=2$f_\#^2 \lambda$ (proportional to phase error)

Geom.Opt: Near field/Far field crossover

From Wright: Image Formation ...

Geom.Opt: Near field/Far field crossover

From Wright: Image Formation ...
Ultrasound imaging

- Near field/far field transition, $D=28\text{mm}$, $f=3.5\text{MHz}$ ⇒
  - $\lambda = 1540 / 3.5 \cdot 10^6 = 0.44\text{mm}$ and $d_R = D^2 / R = 1782\text{mm}$
  - All diagnostic ultrasound imaging occurs in the extreme near field!
- Azimuth resolution, $D=28\text{mm}$, $f=7\text{MHz}$ ⇒
  - $\lambda = 0.22\text{mm}$ and $\theta = \lambda / D = 0.45^{\circ}$,
  - i.e. about 200 lines are required to scan $\pm 45^{\circ}$
- Depth of focus, $f^# = 2$, $f=5\text{MHz}$ ⇒
  - $\lambda = 0.308\text{mm}$ and $\text{DOF} = 2f^2 \lambda \approx 2.5\text{mm}$.
- Ultrasound requires $T = 2 \cdot 2.5 \cdot 10^{-3} / 1540 = 3.2\mu\text{s}$ to travel the DOF. This is the minimum update rate for the delays in a dynamically focused system.

Ambiguities & Aberrations

- Aperture ambiguities
  - Due to symmetries
- Aberrations
  - Deviation in the waveform from its intended form.
  - In optics; due to deviation of a lens from its ideal shape.
  - More generally; Turbulence in the medium, inhomogeneous medium or position errors in the aperture.
  - Ok if small comp. to $\lambda_0$.

Co-array for continuous apertures

- $c(\chi) \equiv \int w(\vec{x})w(\vec{x} + \chi)d\vec{x}$, $\chi$ called lag and its domain lag space.
- Important when array processing algorithms employ the wave’s spatiotemporal correlation function to characterize the wave’s energy.
- Fourier transform of $c(\chi) = |W(\vec{k})|^2$ gives a smoothed estimate of the power spectrum $S_f(k, w)$.