Outline

Finite Continuous Apertures

Spatial sampling
  Sampling in one dimension

Arrays of discrete sensors
  Regular arrays
  Grating lobes
  Element response
  Irregular arrays

Periodic spatial sampling in one dimension

- Array:
  - Consists of individual sensors that sample the environment spatially
  - Each sensor could be an aperture or omni-directional transducer
  - Spatial sampling introduces some complications (Nyquist sampling, folding, ...)

- Question to be asked/answered:
  When can \( f(x, t_0) \) be reconstructed by \( \{y_m(t_0)\} \)?
  - \( f(x, t) \) is the continuous signal and
  - \( \{y_m(t)\} \) is a sequence of temporal signals where \( y_m(t) = f(md, t) \), \( d \) being the spatial sampling interval.

Sampling theorem (Nyquist):

If a continuous-variable signal is band-limited to frequencies below \( k_0 \), then it can be periodically sampled without loss of information so long as the sampling period \( d \leq \pi/k_0 = \lambda_0/2 \).

Figure A.1: The periodic aperture \( f(x) \) is equal to the sum of periodic replicas of the spectrum \( F(k) \). In this case the periodic replicas do not overlap because \( |F(k)| \) is bandlimited to a frequency \( k_0 \). When the aperture \( f(x) \) is not bandlimited to frequencies below \( k_0 \), one period of the periodic spectrum \( F(k) \) then does not equal \( F(k) \). This phenomenon is called aliasing.
Periodic spatial sampling in one dimension ...

- Periodic sampling of one-dimensional signals can be straightforwardly extended to multidimensional signals.
- "Rectangular / regular" sampling not necessary for multidimensional signals.

Regular arrays; linear array

- Consider linear array; $M$ equally spaced ideal sensor with inter-element spacing $d$ along the $x$ direction.
  - The discrete aperture function, $w_m$.
  - The discrete aperture smoothing function, $W(k)$:
    
    $$W(k) \equiv \sum_m w_m e^{jkmd}$$
  - Spatial aliasing given by $d$ relative to $\lambda$.

Grating lobes

- Given an linear array of $M$ sensors with element spacing $d$.
  - $W(k) = \frac{\sin \frac{\pi Md}{2}}{\frac{\pi Md}{2}}$.
  - Mainlobe given by $D = Md$.
  - Grating lobes (if any) given by $d$.
  - Maximal response for $\phi = 0$. Does it exist other $\phi_g$ with the same maximal response?
    
    $$k_x = 2\pi \sin \phi_g \pm 2\pi n \Rightarrow \sin \phi_g = \pm \frac{1}{2} n.$$  
    
    - $n = 1$: No gratinglobes for $\lambda/d > 1$, i.e. $d < \lambda$.
    - $d = 4\lambda$:
      
      $$\sin \phi_g \pm n \cdot 1/4 \Rightarrow \phi_g = \pm 14.5^\circ, \pm 30^\circ, \pm 48.6^\circ, \pm 90^\circ.$$
Element response

- If the elements have finite size:
  \[ W_e(\vec{k}) = \int_{-\infty}^{\infty} w(\vec{k}) e^{i\vec{k} \cdot \vec{x}} d\vec{x} \]

  - If linear array:
    Continuous aperture “devided into” \( M \) parts of size \( d \)
    Each single element: \( \sin(kd/2) \rightarrow \) first zero at \( k = 2\pi/d \)

- Total response:
  \[ W_{\text{total}}(\vec{k}) = W_e(\vec{k}) \cdot W_a(\vec{k}) \]
  where \( W_a(\vec{k}) \) is the array response when point sources are assumed.

Irregular arrays

- Discrete co-array function:
  \[ c(\vec{\chi}) = \sum_{(m_1, m_2) \in \vartheta(\vec{\chi})} w_{m_1} w_{m_2}^* \]
  where \( \vartheta(\vec{\chi}) \) denotes the set of indices \((m_1, m_2)\) for which \( \vec{x}_{m_2} - \vec{x}_{m_1} = \vec{\chi} \).
  \[ 0 \leq c(\vec{\chi}) \leq M = c(\vec{0}) \]
  \[ \Rightarrow \text{sample spacing in the lag-domain must be small enough to avoid aliasing in the spatial power spectrum.} \]
  \[ \text{Redundant lag: The number of distinct baselines of a given length is greater than one.} \]

Examples

- The Haslach array shown on the left has the co-array on the right. Because there are no redundant baselines in the array, the co-array values are all equal to one except at the origin (zero lag), where the co-array value is \( M \).
Irregular arrays

- Sparse arrays
  - Underlying regular grid, all position not filled.
  - Position fills to acquire a given co-array
    - Non-redundant arrays with minimum number of gaps
    - Maximal length redundant arrays with no gaps.
  - Sparse array optimization
    - Irregular arrays can give regular co-arrays ...

Random arrays

- $W(\vec{k}) = \sum_{m=0}^{M-1} e^{j \vec{k} \cdot \vec{x}_m}$ (assumes unity weights)
- $E[W(\vec{k})] = \sum_{m=0}^{M-1} E[e^{j \vec{k} \cdot \vec{x}_m}] = \int p_x(\vec{x}_m) e^{j \vec{k} \cdot \vec{x}_m} d\vec{x} = M \cdot \Phi_x(\vec{k})$
  - i.e. Equals the array pattern of a continuous aperture where the probability density function plays the same role as the weighting function.

- $\text{var}[W(\vec{k})] = E[|W(\vec{k})|^2] - (E[W(\vec{k})])^2$
  - $E[|W(\vec{k})|^2] = E[\sum_{m=0}^{M-1} e^{j \vec{k} \cdot \vec{x}_m} \cdot \sum_{m' \neq m, m'=0}^{M-1} e^{-j \vec{k} \cdot \vec{x}_{m'}}]$
    - $E[M \cdot 1 + \sum_{m, m' \neq m, m'=0}^{M-2} e^{-j \vec{k} \cdot \vec{x}_{m'}}]$
    - Assumes uncorrelated $x_m$ ($E[x \cdot y] = E[x] \cdot E[y]$)
    - $E[|W(\vec{k})|^2] = M + (M^2 - M)|\Phi_x(\vec{k})|^2$
  - $\Rightarrow \text{var}[W(\vec{k})] = M - M|\Phi_x(\vec{k})|^2$

Examples

- Non-redundant arrays == Minimum hole arrays == Golumb arrays 1101, 1100101, 11001000101
- Redundant arrays == Minimum redundancy arrays 1101, 1100101, 1100100101

![Figure 3.27](image.png) A six-sensor filled array and its co-array are shown. Two arrays having the same aperture are derived by successively removing sensors from the array. This thinning procedure results in the depicted co-arrays. To derive the four-sensor perfect array, one must start with a seven-sensor filled array.