Beamspace Adaptive Beamforming and the GSC

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Overview

- The MVDR beamformer: performance and behavior.
- Generalized Sidelobe Canceller reformulation.
- Implementation of the GSC.
- Beamspace interpretation of the GSC.
- Reduced complexity of beamspace MVDR.
Model: Signal, (spatially white) noise, and interference for $M$-element array

$$\tilde{x} = A\tilde{d} + \tilde{n}, \quad E\{\tilde{x}\tilde{x}^H\} = R_s + R_n, \quad R_s = |A|^2 \tilde{d}\tilde{d}^H, \quad R_n = \mathbf{R}_i + \sigma_w^2 \mathbf{I} \quad (1)$$

For spatially white noise only, the DAS beamformer is optimal (in the sense of minimum noise power in output):

$$y_{DAS} = \tilde{w}_{DAS}^H \tilde{x} \text{ for } \tilde{w}_{DAS} = \frac{1}{M} \tilde{d} \quad (2)$$

For spatially non-white interference, the Minimum Variance beamformer minimizes interference-plus-noise power in the beamformer output:

$$\tilde{w}_{mv} = \arg\min_{\tilde{w}} E\left\{\left|\tilde{w}^H \tilde{x}\right|^2\right\} = \arg\min_{\tilde{w}} \tilde{w}^H \mathbf{R} \tilde{w} \quad (3)$$

In other words:

$$E\left\{\left|\tilde{w}_{mv}^H \tilde{n}\right|^2\right\} \leq E\left\{\left|\tilde{w}^H \tilde{n}\right|^2\right\} \text{ for all weight vectors } \tilde{w} \quad (4)$$

Note: This is actually minimum power, not minimum variance. Subtle difference, theoretical equivalence. Exchange $\mathbf{R}$ for $\mathbf{R}_n$...
MV weight vector for the case of one single interfering source with power $\sigma_i^2$ and propagation vector $\vec{d}_i$:

$$\vec{w}_{MV} = \frac{\Lambda M}{\sigma_w^2} \left( \vec{w}_{das,s} - \rho_{si} \frac{M\sigma_i^2}{\sigma_w^2 + M\sigma_i^2} \vec{w}_{das,i} \right)$$

for $\vec{w}_{das,s} = \frac{\vec{v}_s}{M}, \vec{w}_{das,i} = \frac{\vec{v}_i}{M}$ (5)

- **Interpretation:** DAS beamformer steered towards signal minus (scaled) DAS beamformer steered towards interference.
- Scaling depends on INR, i.e. $\frac{\sigma_i^2}{\sigma_w^2}$.

![Graphs showing angular magnitudes in decibels](image)
Constrained minimization is sometimes difficult to implement and analyze.

However: MVDR can be reformulated as unconstrained minimization.

First suggested by Griffiths and Jim (1982) as an alternative implementation of Frosts Linearly Constrained MV (LCMV) beamformer (1972).

This implementation is usually referred to as the \textit{Griffiths-Jim beamformer} or the \textit{GSC}.

Given a matrix $B \in \mathbb{C}^{M,M-1}$ such that $\vec{d}^H B = \vec{0}$. Then the constrained optimization problem:

$$\min_{\vec{w}} \vec{w}^H R \vec{w} \text{ s.t. } \vec{w}^H \vec{d} = 1$$

(6)

is identical to the unconstrained optimization problem:

$$\min_{\vec{\beta}} \left( \frac{1}{M} \vec{d} - B \vec{\beta} \right)^H R \left( \frac{1}{M} \vec{d} - B \vec{\beta} \right)$$

(7)

with solution:

$$\vec{\beta} = \left( B^H RB \right)^{-1} B^H R \frac{\vec{d}}{M}$$

(8)
GSC Implementation

- From now on: Assume $\vec{d} = \vec{1}$, i.e. signal arriving from broadside.
- Can be implemented as a transversal adaptive filter (using e.g. LMS or RLS algorithm).
- Looks like a Wiener filter, in that it subtracts adaptively filtered noise from desired signal...
- ...however, it does not produce MMSE output.
- **Note:** Sensitive to signal-interference correlation, just like MVDR.
Blocking matrix $B$ suggested by Griffiths and Jim:

$$
B = \begin{bmatrix}
1 & -1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & -1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 1 & -1
\end{bmatrix}
$$

(9)

What signals are processed by the lower branch of the GSC?

$$
x_m' = x_m - x_{m+1} \text{ for } m = 0, 1, \cdots, M - 2
$$

(10)

Two-element endfire “beams”. Good for broadband; broadside nulling for all frequencies.
GSC Interpretation: Beamspace

\[ D = \frac{1}{M} \left[ \vec{d}_0, \cdots, \vec{d}_{M-1} \right], \quad \left[ \vec{d}_m \right]_n = e^{j \frac{2\pi m}{M} n} \]  

- Special case: ULA with DFT matrix \( B = D \) without first column \((\vec{d}_0 = \vec{w}_{DAS} \text{ for broadside arrival})\).
- Invertible transformation: \( D^H D = DD^H = \frac{1}{M} I \).

\begin{figure}
\centering
\includegraphics[width=\textwidth]{beampattern.png}
\end{figure}
The constrained optimization problem becomes:

\[
\min_{\vec{w}} E \left\{ \left| (\vec{w}^H \vec{D}^H)(\vec{D}\vec{x}) \right| \right\} \quad \text{s.t.} \quad \vec{w}^H \vec{1} = 1
\]

\[
\Rightarrow \min_{\vec{w}_BS} E \left\{ \left| \vec{w}_{BS}^H \vec{x}_{BS} \right| \right\} = \vec{w}_{BS}^H \vec{R}_{BS} \vec{w}_{BS} \quad \text{s.t.} \quad \vec{w}_{BS} \vec{e}_0 = 1 \quad (12)
\]

Beamspace solution:

\[
\vec{w}_{BS} = \frac{\vec{R}_{BS}^{-1} \vec{e}_0}{\vec{e}_0^H \vec{R}_{BS}^{-1} \vec{e}_0} \quad (13)
\]
Beamspace beamformer output:

\[ y_{BS} = x_{BS,0} + \sum_{m=1}^{M-1} w^*_{BS,m} x_{BS,m} \]  

(14)

- Interpretation is more obvious: DAS beamformer minus weighted set of other DAS beamformers.
- Note that there is only information about the signal in \( x_{BS,0} \).
- Certain beams \( x_{BS,m} \) contain more information about interference than others.
- **Idea:** Remove those \( x_{BS,m} \) that are expected to contain little or no information about interference.
- **Result:** \( \bar{x}_{BS} \) becomes smaller, i.e. \( R_{BS} \) becomes smaller and easier to invert.
- Inversion of \( R \) is \( O(M^3) \).
- Reduced-dimension beamspace presents a way of using a priori knowledge in the adaptive beamformer...
- ...but is this realistic knowledge?
Try $\mathbf{B} = \mathbf{d}_m$, base weights on single-snapshot noise cov. matrix $\mathbf{R}_n = \mathbf{n}\mathbf{n}^H$:

$$\beta = \left( \mathbf{d}_m^H \mathbf{R}_n \mathbf{d}_m \right) \mathbf{d}_m^H \mathbf{R}_n \mathbf{w}_{das} = \frac{y_{p,d} y_{p,d}^*}{|y_{p,d}|^2 + \frac{\sigma_w^2}{M}}$$

(15)

Yields beamspace MV weight vector:

$$\mathbf{w}_{bs} = \mathbf{w}_{das} - \beta \mathbf{d}_m$$

(16)

Yields interference in output:

$$y_{p,bs} = y_{p,\text{das}} \left( 1 - \frac{|y_{p,d}|^2}{|y_{p,d}|^2 + \frac{\sigma_w^2}{M}} \right)$$

(17)

- Is the choice of $\mathbf{d}_m$ arbitrary?
- What is the impact on the white noise gain?
- What is the impact of spatially white noise?
Special case: Adaptive Sidelobe Reduction (from Synthetic Aperture Radar).

Summary: Only use $x_{BS,m}$ for $m = 0, 1, M - 1$ and set

\[ w_{BS,0} = 1, w_{BS,1} = w_{BS,M-1} = \alpha. \]

Complexity reduced from determining $M$ weights to 1 weight.

Additionally, solution is on the form:

\[
[\hat{w}_{ASR}]_m = 1 + \alpha \cos \left( \frac{2\pi m}{M} \right) \tag{18}
\]

which corresponds to a known family of windows (including Hamming and Hann).
Conclusions

- GSC interpretation of MVDR beamformer yields *unconstrained optimization problem*.
- Unconstrained optimization problems are often easier to analyze and implement.
- Beamspace interpretation of GSC can give reduced complexity.
- **Example:** Ultrasound imaging. Reduction from 64-dimensional element space to 3-dimensional beamspace with similar performance. In general: $O(M) \rightarrow O(3)$. 