We consider matrix norms on \((\mathbb{C}^{m,n}, \mathbb{C})\). All results holds for \((\mathbb{R}^{m,n}, \mathbb{R})\).

**Definition 1** (Matrix Norms). A function \(\| \cdot \| : \mathbb{C}^{m,n} \rightarrow \mathbb{C}\) is called a **matrix norm** on \(\mathbb{C}^{m,n}\) if for all \(A, B \in \mathbb{C}^{m,n}\) and all \(\alpha \in \mathbb{C}\)

1. \(\| A \| \geq 0\) with equality if and only if \(A = 0\). \hspace{1cm} \text{(positivity)}
2. \(\| \alpha A \| = |\alpha| \| A \|\). \hspace{1cm} \text{(homogeneity)}
3. \(\| A + B \| \leq \| A \| + \| B \|\). \hspace{1cm} \text{(subadditivity)}

A matrix norm is simply a vector norm on the finite dimensional vector spaces \((\mathbb{C}^{m,n}, \mathbb{C})\) of \(m \times n\) matrices.
Equivalent norms

Adapting some general results on vector norms to matrix norms give

Theorem 2.

1. All matrix norms are equivalent. Thus, if $\| \cdot \|$ and $\| \cdot \|'$ are two matrix norms on $\mathbb{C}^{m,n}$ then there are positive constants $\mu$ and $M$ such that $\mu \| A \| \leq \| A \|' \leq M \| A \|$ holds for all $A \in \mathbb{C}^{m,n}$.

2. A matrix norm is a continuous function $\| \cdot \| : \mathbb{C}^{m,n} \to \mathbb{R}$. 
Submultiplicativity

For matrix norms we usually require that the norm of a product is bounded by the product of the norms. Thus for square matrices $A, B \in \mathbb{C}^{n,n}$ and a matrix norm we most often have the additional property

4. $\|AB\| \leq \|A\|\|B\|$ \hspace{1cm} \text{(submultiplicativity)}.

For a square matrix $A$ and a submultiplicative matrix norm $\|\cdot\|$ we have

$$\|A^k\| \leq \|A\|^k \text{ for } k \in \mathbb{N}. \hspace{1cm} (1)$$
Consistent Matrix norms

When \( m \) and \( n \) vary we have a family of norms which are formally different for each \( m \) and \( n \) since they are defined in different spaces. However, the most common matrix norms are defined by the same formula for all \( m, n \) and we consider mainly such norms.

**Definition 3** (Consistent Matrix Norms). A submultiplicative matrix norm which is defined for all \( m, n \in \mathbb{N} \), is said to be a consistent matrix norm.
The Frobenius Matrix Norm

For $A \in \mathbb{C}^{m,n}$ we define the Frobenius norm by

$$\|A\|_F := \left( \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$ 

$$\|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_n^2} \quad \text{(singular values of } A \text{.)}$$

The Frobenius norm is a consistent matrix norm which is subordinate to the Euclidian vector norm.
Subordinate Matrix Norm

A matrix norm $\| \| \cdot \|_{\alpha}$ on $\mathbb{C}^{m,n}$ is **subordinate** to the vector norms $\| \cdot \|_{\alpha}$ on $\mathbb{C}^{n}$ and $\| \cdot \|_{\beta}$ on $\mathbb{C}^{m}$ if

$$\| A x \|_{\beta} \leq \| A \| \| x \|_{\alpha} \text{ for all } A \in \mathbb{C}^{m,n} \text{ and } x \in \mathbb{C}^{n}.$$
Definition 4. Suppose $m, n \in \mathbb{N}$ are given and let $\| \cdot \|_\alpha$ be a vector norm on $\mathbb{C}^n$ and $\| \cdot \|_\beta$ a vector norm on $\mathbb{C}^m$. For $A \in \mathbb{C}^{m,n}$ we define

$$
\| A \| := \| A \|_{\alpha, \beta} := \max_{x \neq 0} \frac{\| Ax \|_\beta}{\| x \|_\alpha}.
$$

(2)

We call this the $(\alpha, \beta)$ operator norm, the $(\alpha, \beta)$-norm, or simply the $\alpha$-norm if $\alpha = \beta$. 
Operator norm properties

The operator norm has the following properties:

- It is a **matrix norm**
- It is **subordinate** to the vector norms $\| \cdot \|_\alpha$ and $\| \cdot \|_\beta$.
- It is **consistent** if the vector norms $\| \cdot \|_\alpha = \| \cdot \|_\beta$ and they are defined for all $m, n$.
- There is some $x^* \in \mathbb{C}^n$ with $\| x^* \|_\alpha = 1$ such that

$$
\| A \| = \max_{\| x \|_\alpha = 1} \| Ax \|_\beta = \| A x^* \|_\beta.
$$
The $p$ matrix norm

- The operator norms $\| \cdot \|_p$ defined from the $p$-vector norms are of special interest.

- We define

$$\| A \|_p := \max_{x \neq 0} \frac{\| Ax \|_p}{\| x \|_p} = \max_{\| y \|_p = 1} \| Ay \|_p.$$  \hspace{1cm} (3)

- The $p$-norms are consistent matrix norms which are subordinate to the $p$-vector norm.
Explicit expressions

For $A \in \mathbb{C}^{m,n}$ we have:

- $\|A\|_1 = \max_{1 \leq j \leq n} \sum_{k=1}^{m} |a_{k,j}|$
- $\|A\|_2 = \sigma_1$, the largest singular value of $A$
- $\|A\|_\infty = \max_{1 \leq k \leq m} \sum_{j=1}^{n} |a_{k,j}|$

If $A \in \mathbb{C}^{n,n}$ is nonsingular then $\|A^{-1}\|_2 = \frac{1}{\sigma_n}$, the smallest singular value of $A$.

Proof:
Unitary Transformations

An important property of the 2-norm is that it is invariant with respect to unitary transformations.

Let $k, m, n \in \mathbb{N}$, $V \in \mathbb{C}^{k,m}$, $U \in \mathbb{C}^{n,n}$, $A \in \mathbb{C}^{m,n}$, $V^HV = I$ and $U^HU = I$. Then

1. $\|VA\|_2 = \|A\|_2$ and $\|V\|_2 = 1$,
2. $\|AU\|_2 = \|A\|_2$.

Proof:
Example

- $A := \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
- $\|A\|_1 = 6$
- $\|A\|_2 = 5.465$
- $\|A\|_{\infty} = 7.$
- $\|A\|_F = 5.4772$
Perturbation of linear systems

Consider the system of two linear equations

\[ \begin{align*}
x_1 + x_2 &= 20 \\
x_1 + 0.999x_2 &= 19.99
\end{align*} \]

The exact solution is \( x_1 = x_2 = 10 \).

Suppose we replace the second equation by

\[ x_1 + 1.001x_2 = 19.99, \]

the exact solution changes to \( x_1 = 30, x_2 = -10 \).

A small change in one of the coefficients, from 0.999 to 1.001, changed the exact solution by a large amount.
Ill Conditioning

A mathematical problem in which the solution is very sensitive to changes in the data is called **ill-conditioned** or sometimes **ill-posed**.

Such problems are difficult to solve on a computer.

If at all possible, the mathematical model should be changed to obtain a more well-conditioned or properly-posed problem.
Perturbations

We consider what effect a small change (perturbation) in the data $A, b$ has on the solution $x$ of a linear system $Ax = b$.

Suppose $y$ solves $(A + E)y = b + e$ where $E$ is a (small) $n \times n$ matrix and $e$ a (small) vector.

How large can $y - x$ be?

To measure this we use vector and matrix norms.
Conditions on the norms

\[ \| \cdot \| \text{ will denote a vector norm on } \mathbb{C}^n \text{ and also a submultiplicative matrix norm on } \mathbb{C}^{n,n} \text{ which in addition is subordinate to the vector norm.} \]

Thus for any \( A, B \in \mathbb{C}^{n,n} \) and any \( x \in \mathbb{C}^n \) we have

\[ \| AB \| \leq \| A \| \| B \| \text{ and } \| Ax \| \leq \| A \| \| x \|. \]

This is satisfied if the matrix norm is the operator norm corresponding to the given vector norm or the Frobenius norm.
Absolute and relative error

- The difference $\|y - x\|$ measures the absolute error in $y$ as an approximation to $x$,

- $\|y - x\|/\|x\|$ or $\|y - x\|/\|y\|$ is a measure for the relative error.
Perturbation in the right hand side

Theorem 5. Suppose $A \in \mathbb{C}^{n,n}$ is invertible, $b, e \in \mathbb{C}^n$, $b \neq 0$ and $Ax = b$, $Ay = b + e$. Then

$$\frac{1}{K(A)} \frac{\|e\|}{\|b\|} \leq \frac{\|y - x\|}{\|x\|} \leq K(A) \frac{\|e\|}{\|b\|}, \quad K(A) = \|A\| \|A^{-1}\|. \quad (4)$$

Proof:

Consider (4). $\frac{\|e\|}{\|b\|}$ is a measure for the size of the perturbation $e$ relative to the size of $b$. $\frac{\|y - x\|}{\|x\|}$ can in the worst case be

$$K(A) = \|A\| \|A^{-1}\|$$

times as large as $\frac{\|e\|}{\|b\|}$. 
Condition number

$K(A)$ is called the **condition number with respect to inversion of a matrix**, or just the condition number, if it is clear from the context that we are talking about solving linear systems.

The condition number depends on the matrix $A$ and on the norm used. If $K(A)$ is large, $A$ is called **ill-conditioned** (with respect to inversion).

If $K(A)$ is small, $A$ is called **well-conditioned** (with respect to inversion).
Condition number properties

Since \[ \|A\| \|A^{-1}\| \geq \|AA^{-1}\| = \|I\| \geq 1 \] we always have \[ K(A) \geq 1. \]

Since all matrix norms are equivalent, the dependence of \( K(A) \) on the norm chosen is less important than the dependence on \( A \).

Usually one chooses the spectral norm when discussing properties of the condition number, and the \( l_1 \) and \( l_\infty \) norm when one wishes to compute it or estimate it.
The 2-norm

Suppose $A$ has singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$ and eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ if $A$ is square.

$$K_2(A) = \|A\|_2 \|A^{-1}\|_2 = \frac{\sigma_1}{\sigma_n}$$

$$K_2(A) = \|A\|_2 \|A^{-1}\|_2 = \frac{|\lambda_1|}{|\lambda_n|}, \quad A \text{ normal.}$$

It follows that $A$ is ill-conditioned with respect to inversion if and only if $\sigma_1/\sigma_n$ is large, or $|\lambda_1|/|\lambda_n|$ is large when $A$ is normal.

$$K_2(A) = \|A\|_2 \|A^{-1}\|_2 = \frac{\lambda_1}{\lambda_n}, \quad A \text{ positive definite.}$$
Suppose we have computed an approximate solution \( y \) to \( Ax = b \). The vector \( r(y) = Ay - b \) is called the residual vector, or just the residual. We can bound \( x - y \) in terms of \( r(y) \).

**Theorem 6.** Suppose \( A \in \mathbb{C}^{n,n} \), \( b \in \mathbb{C}^{n} \), \( A \) is nonsingular and \( b \neq 0 \). Let \( r(y) = Ay - b \) for each \( y \in \mathbb{C}^{n} \). If \( Ax = b \) then

\[
\frac{1}{K(A)} \frac{\|r(y)\|}{\|b\|} \leq \frac{\|y - x\|}{\|x\|} \leq K(A) \frac{\|r(y)\|}{\|b\|}.
\] (5)
Discussion

- If $A$ is well-conditioned, (5) says that
  \[ \frac{\| y - x \|}{\| x \|} \approx \frac{\| r(y) \|}{\| b \|}. \]

  In other words, the accuracy in $y$ is about the same order of magnitude as the residual as long as $\| b \| \approx 1$.

- If $A$ is ill-conditioned, anything can happen.

- The solution can be inaccurate even if the residual is small.

- We can have an accurate solution even if the residual is large.
We consider next a perturbation in $A$.

**Theorem 7.** Suppose $A, E \in \mathbb{C}^{n,n}$, $b \in \mathbb{C}^n$ with $A$ invertible and $b \neq 0$. If $\|A^{-1}E\| < 1$ for some operator norm then $A + E$ is invertible. If $Ax = b$ and $(A + E)y = b$ then

$$\frac{\|y - x\|}{\|x\|} \leq \frac{\|A^{-1}E\|}{1 - \|A^{-1}E\|} \leq \frac{K(A)}{1 - \|A^{-1}E\| \|A\|} \|E\|.$$  \hspace{1cm} \text{(6)}

- $\|E\|/\|A\|$ is a measure of the size of the perturbation $E$ in $A$ relative to the size of $A$.
- The condition number again plays a crucial role.
The Spectral Radius

- We define the **spectral radius** of a matrix $A \in \mathbb{C}^{n,n}$ as the maximum absolute values of the eigenvalues.

$$\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|.$$  \hfill (7)

- For any submultiplicative matrix norm $\|\cdot\|$ on $\mathbb{C}^{n,n}$ and any $A \in \mathbb{C}^{n,n}$ we have $\rho(A) \leq \|A\|$.

**Proof:**

- Let $A \in \mathbb{C}^{n,n}$ and $\epsilon > 0$ be given. There is a submultiplicative matrix norm $\|\cdot\|$ on $\mathbb{C}^{n,n}$ such that $\rho(A) \leq \|A\| \leq \rho(A) + \epsilon$.

**Proof:**
Limits

For any $A \in \mathbb{C}^{n,n}$ we have

$$\lim_{k \to \infty} A^k = 0 \iff \rho(A) < 1.$$  

Convergence can be slow:

$$A = \begin{bmatrix} 0.99 & 1 & 0 \\ 0 & 0.99 & 1 \\ 0 & 0 & 0.99 \end{bmatrix}, \quad A^{100} = \begin{bmatrix} 0.4 & 9.37 & 1849 \\ 0 & 0.4 & 37 \\ 0 & 0 & 0.4 \end{bmatrix},$$  

$$A^{2000} = \begin{bmatrix} 10^{-9} & \epsilon & 0.004 \\ 0 & 10^{-9} & \epsilon \\ 0 & 0 & 10^{-9} \end{bmatrix}.$$