UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in INF-MAT 4350 — Numerical linear algebra

Day of examination: 4 December 2008

Examination hours: 0900 – 1200

This problem set consists of 4 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 7 part questions will be weighted equally.

Problem 1 Iterative methods

Consider the linear system $Ax = b$ in which

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 7 & 2 \\ 1 & 2 & 4 \end{pmatrix},$$

and $b = (1, 9, -2)^T$.

1a

With $x_0 = (1, 1, 1)^T$, carry out one iteration of the Gauss-Seidel method to find $x_1 \in \mathbb{R}^3$.

Answer:

$$\begin{aligned} x_1(1) &= (b(1) - x_0(3))/3 = (1-1)/3 = 0, \\ x_1(2) &= (b(2) - 2x_0(3))/7 = (9 - 2*1)/7 = 1, \\ x_1(3) &= (b(3) - x_1(1) - 2x_1(2))/4 = (-2 - 0 - 2*1)/4 = -1. \end{aligned}$$

1b

If we continue the iteration, will the method converge? Why?

Answer: Yes. Since A is diagonally dominant, i.e.,

$$a_{ii} > \sum_{j \neq i} |a_{ij}|, \qquad i = 1, 2, 3,$$

(Continued on page 2.)

all its eigenvalues are positive by Gerschgorin's circle theorem. Therefore, since A is also symmetric it is positive definite, which guarantees convergence.

1c

Write a matlab program for the Gauss-Seidel method applied to a matrix $A \in \mathbb{R}^{n,n}$ and right-hand side $b \in \mathbb{R}^n$. Use the ratio of the current residual to the initial residual as the stopping criterion, as well as a maximum number of iterations.

Hint: The function C=tril(A) extracts the lower part of A into a lower triangular matrix C.

Answer:

```
function [x,it]=gs(A,b,x,tol,maxit)
nr=norm(b-A*x,2);
C = tril(A);
for k=1:maxit
  r=b-A*x;
  x=x+C\r;
  if norm(r,2)/nr<tol
    it=k; return;
  end
end
it = maxit;
```

Problem 2 QR factorization

Let

$$A = \begin{pmatrix} 2 & 1 \\ 2 & -3 \\ -2 & -1 \\ -2 & 3 \end{pmatrix}.$$

2a

Find the Cholesky factorization of $A^T A$.

Answer:

$$A^T A = \begin{pmatrix} 16 & -8 \\ -8 & 20 \end{pmatrix}.$$

We want $A^T A = R^T R$ where

$$R = \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix},$$

and $r_{11}, r_{22} > 0$. Since

$$R^T R = \begin{pmatrix} r_{11}^2 & r_{11} r_{12} \\ r_{11} r_{12} & r_{12}^2 + r_{22}^2 \end{pmatrix},$$

(Continued on page 3.)

we need

$$r_{11}^2 = 16, \qquad r_{11}r_{12} = -8, \qquad r_{12}^2 + r_{22}^2 = 20.$$

The solution is $r_{11} = 4$, $r_{12} = -2$, $r_{22} = 4$.

2b

Find the QR factorization of A.

Answer: We have already found R:

$$R = \begin{pmatrix} 4 & -2 \\ 0 & 4 \end{pmatrix}.$$

Then, to get $A = QR$ we need $Q = AR^{-1}$ and since

$$R^{-1} = \frac{1}{16}\begin{pmatrix} 4 & 2 \\ 0 & 4 \end{pmatrix},$$

we find

$$Q = AR^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Problem 3 Kronecker products

Let $A, B \in \mathbb{R}^{n,n}$. Show that the eigenvalues of the Kronecker product $A \otimes B$ are products of the eigenvalues of A and B and that the eigenvectors of $A \otimes B$ are Kronecker products of the eigenvectors of A and B.

Answer: Suppose that $Au = \lambda u$ and $Bv = \mu v$. Then

$$\begin{aligned}
(A \otimes B)(u \otimes v) &= \begin{pmatrix} Ab_{11} & \cdots & Ab_{1n} \\ \vdots & & \vdots \\ Ab_{n1} & \cdots & Ab_{nn} \end{pmatrix} \begin{pmatrix} uv_1 \\ \vdots \\ uv_n \end{pmatrix} \\
&= \begin{pmatrix} Ab_{11}uv_1 + \cdots + Ab_{1n}uv_n \\ \vdots \\ Ab_{n1}uv_1 + \cdots + Ab_{nn}uv_n \end{pmatrix} = \begin{pmatrix} Au(Bv)_1 \\ \vdots \\ Au(Bv)_n \end{pmatrix} \\
&= (Au) \otimes (Bv) = (\lambda u) \otimes (\mu v) = (\lambda\mu)(u \otimes v).
\end{aligned}$$

Problem 4 Matrix norms

Suppose $A \in \mathbb{R}^{n,n}$ is invertible, $b, c \in \mathbb{R}^n$, $b \neq 0$, and $Ax = b$ and $Ay = b + e$. Show that

$$\frac{1}{K(A)}\frac{\|e\|}{\|b\|} \le \frac{\|y - x\|}{\|x\|} \le K(A)\frac{\|e\|}{\|b\|},$$

(Continued on page 4.)

where $\|\cdot\|$ is the Euclidean vector norm in $\mathbb{R}^n$ and $K(A)$ is the condition number of A with respect to the matrix 2-norm.

Answer: Since $Ax = b$, we have

$$\|b\| = \|Ax\| \leq \|A\|\|x\| \tag{1}$$

and because $A^{-1}b = x$, we have

$$\|x\| = \|A^{-1}b\| \leq \|A^{-1}\|\|b\| \tag{2}$$

Similarly, since $A(y - x) = e$, we have

$$\|e\| = \|A(y - x)\| \leq \|A\|\|y - x\| \tag{3}$$

and because $A^{-1}e = y - x$, we have

$$\|y - x\| = \|A^{-1}e\| \leq \|A^{-1}\|\|e\|. \tag{4}$$

Now, since $K(A) = \|A\|\|A^{-1}\|$, equations (2) and (3) give the lower bound on $\|y - x\|/\|x\|$ and equations (1) and (4) give the upper bound.

Good luck!