UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in INF-MAT 3350/4350 — Numerical linear algebra

Day of examination: 6 December 2007

Examination hours: 0900 – 1200

This problem set consists of 2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 9 part questions will be weighted equally.

Problem 1 Orthogonal transformations

A Householder transformation is a matrix $H \in \mathbb{R}^{n,n}$ of the form

$$H = I - uu^T,$$

where $u \in \mathbb{R}^n$ is such that $u^T u = 2$.

1a

Show that H is symmetric.

1b

Show that H is an orthogonal transformation.

1c

Find Hx when

$$u = \frac{1}{\sqrt{10}} \begin{pmatrix} -2 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

1d

Find a QR-factorization of the matrix

$$A = \begin{pmatrix} 3 & 5 \\ 4 & 10 \end{pmatrix}.$$

(Continued on page 2.)

Problem 2 Eigenvalues

2a

Suppose the matrix $A \in \mathbb{R}^{n,n}$ has linearly independent eigenvectors $v_1, \ldots, v_n$ and that its eigenvalues are such that $\lambda_1 > |\lambda_2| \geq \cdots \geq |\lambda_n|$. If $z_0 \in \mathbb{R}^n$ is a vector such that $v_1^T z_0 \neq 0$, and we let $z_k = A^k z_0$ and $x_k = z_k/\|z_k\|$, what does x_k converge to as $k \to \infty$?

2b

If u is an approximation to an eigenvector of $A \in \mathbb{R}^{n,n}$ then an approximation to the corresponding eigenvalue is the value λ that minimizes the function

$$\rho(\lambda) = \|Au - \lambda u\|_2.$$

Find λ which minimizes ρ.

2c

What is the QR-algorithm for finding all eigenvalues of a real matrix $A \in \mathbb{R}^{n,n}$, assuming they are all real and distinct? (You do not need to discuss convergence conditions).

Problem 3 Iterative methods

The Jacobi method for solving the linear system $Ax = b$ is

$$x_{k+1} = x_k + D^{-1} r_k, \tag{1}$$

where $r_k = b - Ax_k$, and D is the diagonal matrix with $d_{ii} = a_{ii}$.

3a

By writing (1) in the form

$$x_{k+1} = Bx_k + c, \tag{2}$$

for some B and c, derive a condition on $\rho(B)$ that guarantees the convergence of (1).

3b

By applying Gerschgorin's circle theorem to B, derive a condition on A that guarantees the convergence of (1).

Good luck!