UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in INF-MAT 3350/4350 — Numerical linear algebra

Day of examination: 6 December 2007

Examination hours: 0900 – 1200

This problem set consists of 4 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 9 part questions will be weighted equally.

Problem 1 Orthogonal transformations

A Householder transformation is a matrix $H \in \mathbb{R}^{n,n}$ of the form

$$H = I - uu^T,$$

where $u \in \mathbb{R}^n$ is such that $u^T u = 2$.

1a

Show that H is symmetric.

Answer:

$$H^T = I^T - (uu^T)^T = I - (u^T)^T u^T = H.$$

1b

Show that H is an orthogonal transformation.

Answer: Must show that $H^T H = I$. By symmetry, enough to show that $H^2 = I$:

$$H^2 = (I - uu^T)(I - uu^T) = I - 2uu^T + u(u^T u)u^T = I,$$

because $u^T u = 2$.

(Continued on page 2.)

1c

Find Hx when

$$u = \frac{1}{\sqrt{10}}\begin{pmatrix}-2\\4\end{pmatrix} \qquad \text{and} \qquad x = \begin{pmatrix}3\\4\end{pmatrix}.$$

Answer:

Since

$$uu^T = \frac{1}{10}\begin{pmatrix}4 & -8\\-8 & 16\end{pmatrix} = \begin{pmatrix}2/5 & -4/5\\-4/5 & 8/5\end{pmatrix},$$

we find that

$$H = \begin{pmatrix}3/5 & 4/5\\4/5 & -3/5\end{pmatrix}.$$

Then

$$Hx = \begin{pmatrix}3/5 & 4/5\\4/5 & -3/5\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}5\\0\end{pmatrix}.$$

1d

Find a QR-factorization of the matrix

$$A = \begin{pmatrix}3 & 5\\4 & 10\end{pmatrix}.$$

Answer:

Since x is the first column of A and $Hx = 5e_1$, we can simply use H to obtain Q and R in one step. We find

$$HA = \begin{pmatrix}5 & 11\\0 & -2\end{pmatrix},$$

and therefore $A = QR$ where

$$Q = H^{-1} = H = \begin{pmatrix}3/5 & 4/5\\4/5 & -3/5\end{pmatrix} \qquad \text{and} \qquad R = \begin{pmatrix}5 & 11\\0 & -2\end{pmatrix}.$$

Problem 2 Eigenvalues

2a

Suppose the matrix $A \in \mathbb{R}^{n,n}$ has linearly independent eigenvectors $v_1, \dots, v_n$ and that its eigenvalues are such that $\lambda_1 > |\lambda_2| \geq \cdots \geq |\lambda_n|$. If $z_0 \in \mathbb{R}^n$ is a vector such that $v_1^T z_0 \neq 0$, and we let $z_k = A^k z_0$ and $x_k = z_k/\|z_k\|$, what does x_k converge to as $k \to \infty$?

Answer:

We can express z_0 as

$$z_0 = c_1 v_1 + \cdots + c_n v_n,$$

(Continued on page 3.)

where $c_1 \neq 0$ because $v_1^T z_0 \neq 0$. Then, since

$$z_k = c_1 \lambda_1^k v_1 + \cdots + c_n \lambda_n^k v_n,$$

we have, if $c_1 > 0$,

$$x_k = \frac{v_1 + (c_2/c_1)(\lambda_2/\lambda_1)^k v_2 + \cdots (c_n/c_1)(\lambda_n/\lambda_1)^k v_n}{\|v_1 + (c_2/c_1)(\lambda_2/\lambda_1)^k v_2 + \cdots (c_n/c_1)(\lambda_n/\lambda_1)^k v_n\|} \to \frac{v_1}{\|v_1\|},$$

as $k \to \infty$, and, if $c_1 < 0$,

$$x_k = \frac{-v_1 - (c_2/c_1)(\lambda_2/\lambda_1)^k v_2 - \cdots - (c_n/c_1)(\lambda_n/\lambda_1)^k v_n}{\|v_1 + (c_2/c_1)(\lambda_2/\lambda_1)^k v_2 + \cdots (c_n/c_1)(\lambda_n/\lambda_1)^k v_n\|} \to \frac{-v_1}{\|v_1\|},$$

as $k \to \infty$.

2b

If u is an approximation to an eigenvector of $A \in \mathbb{R}^{n,n}$ then an approximation to the corresponding eigenvalue is the value λ that minimizes the function

$$\rho(\lambda) = \|Au - \lambda u\|_2.$$

Find λ which minimizes ρ.

Answer:
It is sufficient to minimize $E(\lambda) = \rho^2(\lambda)$, and

$$E(\lambda) = (Au)^T(Au) - 2u^T(Au)\lambda + u^T u \lambda^2.$$

We see that E is a quadratic polynomial and since $u^T u = \|u\|_2^2 > 0$, E has a unique minimum, where $E'(\lambda) = 0$. Since

$$E'(\lambda) = -2u^T Au + 2u^T u\lambda,$$

we thus obtain the minimum when

$$\lambda = \frac{u^T Au}{u^T u}.$$

2c

What is the QR-algorithm for finding all eigenvalues of a real matrix $A \in \mathbb{R}^{n,n}$, assuming they are all real and distinct? (You do not need to discuss convergence conditions).

Answer:
(i) $A_1 = A$. (ii) For $k = 1, 2, \ldots$, find the QR-factorization of A_k, i.e., $Q_k R_k = A_k$, and set $A_{k+1} = R_k Q_k$.

The elements of A_k below the diagonal converge to 0 and elements on the diagonal of A_k converge to the eigenvalues of A.

(Continued on page 4.)

Problem 3 Iterative methods

The Jacobi method for solving the linear system $Ax = b$ is

$$x_{k+1} = x_k + D^{-1} r_k, \tag{1}$$

where $r_k = b - Ax_k$, and D is the diagonal matrix with $d_{ii} = a_{ii}$.

3a

By writing (1) in the form

$$x_{k+1} = Bx_k + c, \tag{2}$$

for some B and c, derive a condition on $\rho(B)$ that guarantees the convergence of (1).

Answer: $B = I - D^{-1}A$ and $c = D^{-1}b$. Since (1) holds when x_k is replaced by x, so does (2), i.e.,

$$x = Bx + c,$$

and substracting this from (2) and defining $e_k = x_k - x$ gives

$$e_{k+1} = Be_k.$$

Therefore, for any vector norm, $\|\cdot\|$, $\|e_{k+1}\| \le \|B\|\|e_k\|$, and so $\|e_k\| \le \|B\|^k\|e_0\|$, and the method converges if $\|B\| < 1$. So a sufficient condition for convergence is that $\rho(B) < 1$, i.e., that all eigenvalues of B are less than one in absolute value.

3b

By applying Gerschgorin's circle theorem to B, derive a condition on A that guarantees the convergence of (1).

Answer: By Gerschgorins circle theorem, all eigenvalues of B are in the union of the discs $B(b_{ii}, r_i)$, where

$$r_i = \sum_{j \neq i} |b_{ij}|.$$

Therefore, since $b_{ii} = 0$, a sufficient condition for the convergence of (1) is that $r_i < 1$ for all i. Since $b_{ij} = a_{ij}/a_{ii}$ for $i \neq j$, this condition is equivalent to the condition

$$\sum_{j \neq i} |a_{ij}| < |a_{ii}|,$$

i.e., that A is strictly diagonally dominant.

Good luck!