UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in INF-MAT4350 — Numerical linear algebra

Day of examination: 3 December 2009

Examination hours: 0900 – 1200

This problem set consists of 4 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 7 part questions will be weighted equally.

Problem 1 Matrix products

Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{E} \in \mathbb{R}^{n,n}$ be matrices where $\boldsymbol{A}^T = \boldsymbol{A}$. In this problem an (arithmetic) operation is an addition or a multiplication. We ask about exact numbers of operations.

1a

How many operations are required to compute the matrix product $\boldsymbol{BC}$? How many operations are required if $\boldsymbol{B}$ is lower triangular?

Answer: For each of the n^2 elements in $\boldsymbol{B}$ we have to compute an inner product of length n. This requires n multiplications and $n-1$ additions. Therefore to compute $\boldsymbol{BC}$ requires $n^2(2n-1) = 2n^3 - n^2$ operations.

If $\boldsymbol{B}$ is lower triangular then row k of $\boldsymbol{B}$ contains k non-zero elements, $k = 1, \dots, n$. Therefore, to compute an element in the k-th row of $\boldsymbol{BC}$ requires k multiplications and $k-1$ additions. Hence in total we need $n\sum_{k=1}^{n}(2k-1) = n^3$ operations.

1b

Show that there exists a lower triangular matrix $\boldsymbol{L} \in \mathbb{R}^{n,n}$ such that $\boldsymbol{A} = \boldsymbol{L} + \boldsymbol{L}^T$.

Answer: We have $\boldsymbol{A} = \boldsymbol{A_L} + \boldsymbol{A_D} + \boldsymbol{A_R}$, where $\boldsymbol{A_L}$ is lower triangular with 0 on the diagonal, $\boldsymbol{A_D} = \operatorname{diag}(a_{11}, \dots, a_{nn})$, and $\boldsymbol{A_R}$ is upper triangular with 0 on the diagonal. Since $\boldsymbol{A}^T = \boldsymbol{A}$, we have $\boldsymbol{A_R} = \boldsymbol{A_L}^T$. If we let $\boldsymbol{L} := \boldsymbol{A_L} + \frac{1}{2}\boldsymbol{A_D}$ we obtain $\boldsymbol{A} = \boldsymbol{L} + \boldsymbol{L}^T$.

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1c

We have $\boldsymbol{E}^T\boldsymbol{A}\boldsymbol{E} = \boldsymbol{S} + \boldsymbol{S}^T$ where $\boldsymbol{S} = \boldsymbol{E}^T\boldsymbol{L}\boldsymbol{E}$. How many operations are required to compute $\boldsymbol{E}^T\boldsymbol{A}\boldsymbol{E}$ in this way?

Answer: **Svar**: We need n operations to compute the diagonal in $\boldsymbol{L}$. From Question (1a) we need n^3 operations to compute $\boldsymbol{L}\boldsymbol{E}$ and consequently $2n^3 - n^2$ operations to compute $\boldsymbol{E}^T(\boldsymbol{L}\boldsymbol{E})$. Therefore n^2 operations to compute the sum $\boldsymbol{S} + \boldsymbol{S}^T$. In total $3n^3 + n$ operations. Direct computation of $\boldsymbol{E}^T\boldsymbol{A}\boldsymbol{E}$ requires $4n^3 - 2n^2$ operations.

Problem 2 Gershgorin Disks

The eigenvalues of $\boldsymbol{A} \in \mathbb{R}^{n,n}$ lie inside $R \cap C$, where $R := R_1 \cup \cdots \cup R_n$ is the union of the row disks R_i of $\boldsymbol{A}$, and $C = C_1 \cup \cdots \cup C_n$ is the union of the column disks C_j. You do not need to prove this. Write a Matlab function `[s,r,c]=gershgorin(A)` that computes the centres $\boldsymbol{s} = [s_1, \ldots, s_n] \in \mathbb{R}^n$ of the row and column disks, and their radii $\boldsymbol{r} = [r_1, \ldots, r_n] \in \mathbb{R}^n$ and $\boldsymbol{c} = [c_1, \ldots, c_n] \in \mathbb{R}^n$, respectively.

Answer:
With for-loops:

```
function [s,r,c] = gershgorin(A)
n=length(A);
s=diag(A); r=zeros(n,1); c=r;
for i=1:n
    for j=1:n
        r(i)=r(i)+abs(A(i,j));
        c(i)=c(i)+abs(A(j,i));
    end;
    r(i)=r(i)-abs(s(i)); c(i)=c(i)-abs(s(i));
end
```

Vectorized:

```
function [s,r,c] = gershgorinv(A)
n=length(A);
s=diag(A); e=ones(n,1);
r=abs(A)*e-abs(s);
c=(abs(A))'*e-abs(s);
```

Problem 3 Eigenpairs

Let $\boldsymbol{A} \in \mathbb{R}^{n,n}$ be tridiagonal (i.e. $a_{ij} = 0$ when $|i - j| > 1$) and suppose also that $a_{i+1,i}a_{i,i+1} > 0$ for $i = 1, \ldots, n-1$.

3a

Show that for an arbitrary nonsingular diagonal matrix $\boldsymbol{D} = \mathrm{diag}(d_1, d_2, \ldots, d_n) \in \mathbb{R}^{n,n}$, the matrix

$$\boldsymbol{B} = \boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D} \tag{1}$$

is tridiagonal by finding a formula for b_{ij}, $i, j = 1, \ldots, n$.

Answer: If $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{D}$ then

$$c_{ij} = \sum_{k=1}^{n} a_{ik}(\boldsymbol{D})_{kj} = a_{ij}(\boldsymbol{D})_{jj} = a_{ij}d_j,$$

and

$$b_{ij} = \sum_{k=1}^{n} (\boldsymbol{D}^{-1})_{ik}c_{kj} = d_i^{-1}c_{ij} = d_i^{-1}a_{ij}d_j.$$

This shows that $b_{ij} = 0$ when $|i - j| > 1$.

3b

Show that there exists a choice of $\boldsymbol{D}$ such that $\boldsymbol{B}$ is symmetric and determine b_{ii} for $i = 1, \ldots, n$ and $b_{i,i+1}$ for $i = 1, \ldots, n-1$ with the choice $d_1 = 1$.

Answer:

$$\begin{aligned}
&\boldsymbol{B} \text{ symmetric} \\
&\quad \Leftrightarrow b_{i,i+1} = b_{i+1,i}, \quad i = 1, \ldots, n-1 \\
&\quad \Leftrightarrow d_i^{-1}a_{i,i+1}d_{i+1} = d_{i+1}^{-1}a_{i+1,i}d_i, \quad i = 1, \ldots, n-1 \\
&\quad \Leftrightarrow \frac{d_{i+1}^2}{d_i^2} = \frac{a_{i+1,i}}{a_{i,i+1}} =: \alpha_i, \quad i = 1, \ldots, n-1 \\
&\quad \Leftrightarrow \frac{d_{i+1}}{d_i} = \pm\sqrt{\alpha_i}, \quad i = 1, \ldots, n-1 \\
&\quad \Leftrightarrow d_{i+1} = \pm d_i\sqrt{\alpha_i}, \quad i = 1, \ldots, n-1.
\end{aligned}$$

So $\boldsymbol{B}$ will be symmetric if we choose

$$d_1 = 1, \quad \text{and} \quad d_{i+1} = d_i\sqrt{\alpha_i}, \quad i = 1, \ldots, n-1.$$

We find $b_{ii} = d_i^{-1}a_{ii}d_i = a_{ii}$ for $i = 1, \ldots, n$ and

$$b_{i,i+1} = d_i^{-1}a_{i,i+1}d_{i+1} = a_{i,i+1}\sqrt{\alpha_i} = \mathrm{sign}(a_{i,i+1})\sqrt{a_{i,i+1}a_{i+1,i}}, \quad i = 1, \ldots, n-1.$$

3c

Show that $\boldsymbol{B}$ and $\boldsymbol{A}$ have the same characteristic polynomials and explain why $\boldsymbol{A}$ has real eigenvalues.

Answer:

$$\begin{aligned}
\pi_{\boldsymbol{B}}(\lambda) &= \det(\boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D} - \lambda\boldsymbol{I}) = \det\left(\boldsymbol{D}^{-1}(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{D}\right) \\
&= \det(\boldsymbol{D}^{-1})\det(\boldsymbol{A} - \lambda\boldsymbol{I})\det(\boldsymbol{D}) = \det(\boldsymbol{D}^{-1}\boldsymbol{D})\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \pi_{\boldsymbol{A}}(\lambda).
\end{aligned}$$

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Since $\boldsymbol{B}$ is real and symmetric it has real eigenvalues and since $\pi_{\boldsymbol{B}} = \pi_{\boldsymbol{A}}$, $\boldsymbol{A}$ has the same real eigenvalues as $\boldsymbol{B}$.

Good luck!