Dynamic logic for Hybrid systems

Differential Dynamic Logic for Verifying Parametric Hybrid Systems

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An example of a hybrid system: Thermostat

A room is heated by a radiator controlled by a thermostat.

When the radiator is off, the temperature decreases (over time) according to the differential equation:

$$\dot{y} = -ay$$

When the radiator is on, the temperature increases towards 30:

$$\dot{y} = -a(y - 30)$$

Solution: Solve the equation...

$$\dot{y} = -ay, \qquad \dot{y} + ay = 0$$

$$e^{at}\,\dot{y} + e^{at}ay = 0, \qquad e^{at}\,\dot{y} + e^{at}y = 0$$

$$(e^{at}y)' = 0, \qquad \int (e^{at}y)' = \int 0 + C$$

$$e^{at}y = C, \qquad y = Ce^{-at}$$

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The other equation is $y(t) = 30a - De^{-at}$

Can we combine the two equations in this way:?

$y(t) = Ce^{-at}$ if $y(t) > 21$

$y(t) = 30a - De^{-at}$ if $y(t) < 19$

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No, we want to express system properties like:

- If the temperature rises above 21, then the raditator should be switched off
- If the temperature falls below 19, then the raditator should be switched on
- We accept a small error margin of one degree. (Radiator *may* switch off at 21 degrees, *must* switch off at 22).

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Hybrid system

According to Wikipedia, the accurate source of all wisdom: *A hybrid system is a dynamic system that exhibits both continuous and discrete dynamic behavior*

Modelling and analysis of hybrid systems combines techniques from mathematics (behavior) and computer science (statefulness, nondeterminism).

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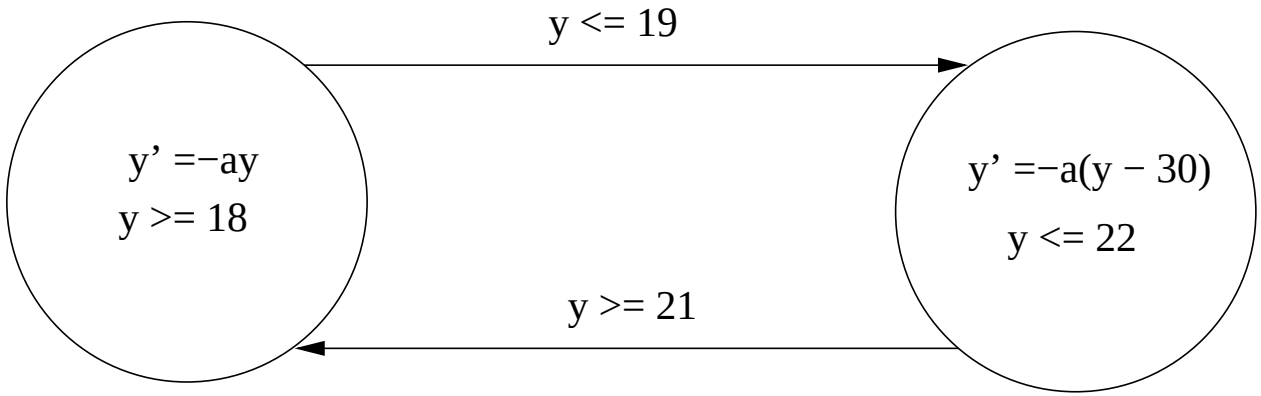
The traditional way of formally modelling hybrid systems is through hybrid automata:



In this paper, however, a state-transition system is used, combined with a dynamic logic.

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d$\mathcal{L}$

The *differential dynamic logic* is a first-order dynamic logic. Its programs are *hybrid programs*. The definition of the hybrid programs motivates the semantics and verification calculus of d$\mathcal{L}$

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Hybrid programs

- V is the set of variables, such as x.
- $Trm(V)$ is the set of terms - *polynomials*, for example $x^2 + y + c$.
- $Fml(V)$ is the set of formulas, such as $x > 0 \wedge x < 42$.

Hybrid programs consist of:

- Discrete jumps (variable resets): $x := \theta$, where x is a variable and θ a term.
- Continuous evolutions: $\dot{x}= \theta \& \chi$, where χ defines the region the evolution must remain in.
- $\alpha; \gamma$, $\alpha \cup \gamma$, α^* and $?\chi$ represent sequential composition, nondeterministic choice, nondeterministic repetition and condition.

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Modalities

We want to relate programs, α, with requirements, ϕ:

We write: $[\alpha]\phi$

This means that: ϕ must hold after every terminating execution of the program α.

($< \alpha > \phi$ means that ϕ must hold after at least one (terminating) execution of α)

Example program

$$[x := 0; \dot{x}= 2](x \geq 0 \land x \leq 42)$$

First, x is set to 0. Then x evolves continuously according to $\dot{x}= 2$. Does the property hold?

States

The variables will have certain values at every point in our program - that's the *state*. For example: $\nu = \{x = 42, y = 666\}$.

- $Sta(V)$ is the set of all states.
- $val(\nu, x)$ gives us the value of the variables x in state ν.
  $val(\nu, x) = 42$

State transitions

A state transition is a relation between a state before the execution of a program, and a state after the execution of a program.

$(x = 42, x = 0)$ is a state transition.

The set of all state transitions of a program α is called the *semantics* of that program, $\rho(\alpha)$.

$(x = 42, x = 0) \in \rho(x := 0)$

$(x = 666, x = 0) \in \rho(x := 0)$

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Example program

A Formula 1 racing car (x) does 0-100 km/h in 3.7 seconds, a while a Lada (y) does it, eventually, in 11.2 seconds. Assuming a constant acceleration, how fast are they going after 10 seconds (τ)?

$$\tau := 0; x := 0; y := 0;$$

$$((?\tau < 10); \dot{x}{=}\ 27, \dot{y}{=}\ 8.9, \dot{\tau}{=}\ 1 \& \tau \leq 10)^*; (?\tau >= 10)$$

We usually rewrite this to:

$$\tau := 0; x := 0; y := 0;$$

$$\mathit{while}\ (?\tau < 10)\ \mathit{do}\ \ \dot{x}{=}\ 27, \dot{y}{=}\ 8.9, \dot{\tau}{=}\ 1 \& \tau \leq 10$$

We know the velocities at time 0 (0 km/h!), but we want to know the function that computes it at time 10.

We can construct those functions easily. Why? Because we know the derivatives of the functions! If $\dot{x}= 27$, then $x(t) = 27t + C$. The constant C is the function's value at time 0: $x(t) = 27t + 0$

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Thus we have:

- $x(t) = 27t$
- $y(t) = 8.9t$
- $\tau(t) = t$

The function x, y and τ are, of course, continuous on the interval $(0, 10)$, and the derivative never changes from the formula specified.

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How can we relate these functions to states and transitions?

Like this:

Start state : $\{x = 0, y = 0, \tau = 0\} = \{x(0), y(0), \tau(0)\}$

We also have a state for each value in the $(0, 10)$ interval:

$\{x(1), y(1), \tau(1)\} = \{x = 27, y = 8.9, \tau = 1\}$

$\{x(2), y(2), \tau(2)\} = \{x = 54, y = 17.8, \tau = 2\}$

$\{x(3.14), y(3.14), \tau(3.14)\} = \{x = 84.78, y = 27.94, \tau = 3.14\}$

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Let's define a function f. It takes a number as input, and produces a desirable state as output:

$$f(t) = \{x(t), y(t), \tau(t)\}, t \in [0, 10]$$

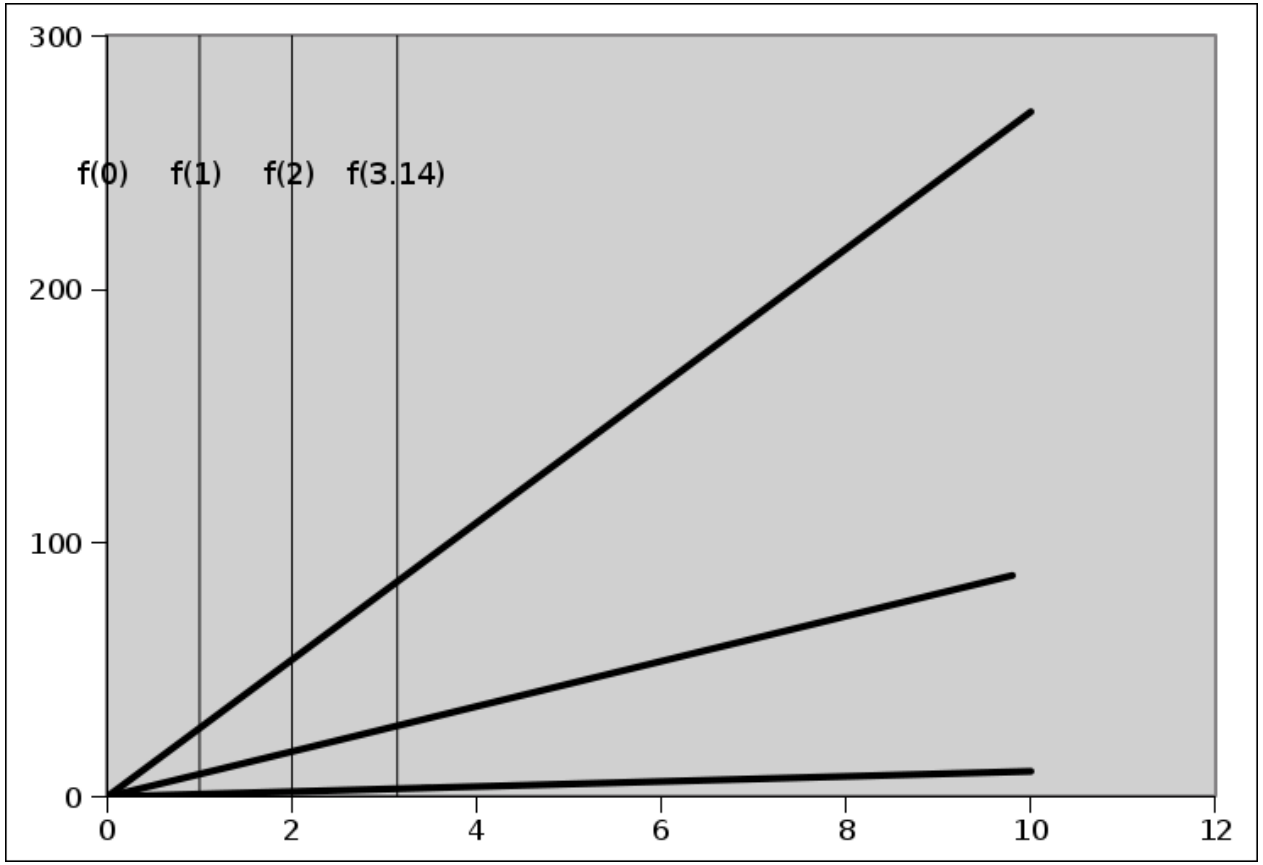
We have, because the *val* function gives us the value of a variable in a state, the following:

$val(f(1), x) = x(1) = 27$

$val(f(2), y) = y(2) = 17.8$

$val(f(3.14), \tau) = \tau(3.14) = 3.14$

Illustration of continuous evolution



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Semantics: Continuous evolution

$(\nu, \omega) \in \rho(\dot{x} = \theta \& \chi) :\Leftrightarrow$ there is a function $f : [0, t] \to Sta(V), t \geq 0$ such that:

- $f(0) = \nu, f(t) = \omega$
- $val(f(\zeta), x)$ is continuous in ζ on $[0, t]$
- $val(f(\zeta), \theta)$ is the derivative of f at each time $\zeta \in (0, t)$.
- For $y \neq x$ and $\zeta \in (0, t)$ we have that $val(f(\zeta), y) = val(\nu, y)$
- $val(f(\zeta), \chi) = true$ for each $\zeta \in (0, t)$

If we have several differential equations then they they are treated accordingly.

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Semantics (simplified!)

- $x := d$, $(\nu, \nu[x \mapsto d])$
- $\chi?$, (ν, ν) if χ
- $\alpha; \gamma$, $(\nu, z); (z, \omega)$ gives (ν, ω)
- $\alpha \cup \gamma$, $\rho(\alpha) \cup \rho(\gamma)$
- α^*, $(\nu_0, \nu_1); \ldots; (\nu_i, \nu_{i+1})$ gives $(\nu_i, \nu_{i+1}), 0 \leq i$

Example: 'Speed supervision in Train Control'

Part of the safety measures of trains is that they must be told how much further they can move safely.

A train driver must then consider something like: 'I have driven 34 km, I can drive until the 35 km mark, and my speed is 120 km/h. When I hit the brakes, the speed decreases at 1 km/h per second. Am I able to stop?'

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Example: 'Speed supervision in Train Control'

- Let m be the 'mark' (35000 m)
- Let z be the 'current position' (34000 m)
- Let s be the safety margin (500 m)
- Let ν be the speed (35 m/s)
- Let b be the brake force (1 m/s^2)
- Let a be the (de)acceleration
- Let τ be the time since the driver last considered his options
- Let ϵ be maximal time the driver may spend between considerations (5 s)

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Hybrid program

$$\psi \rightarrow [(corr; drive)^*]z \leq m$$

$$corr \equiv (?m - z < s; a := -b) \cup (?m - z \geq s; a := 0)$$

$$drive \equiv \tau := 0; (\dot{z} = v, \dot{v} = a, \dot{\tau} = 1 \& v \geq 0 \wedge \tau \leq \epsilon)$$

We will examine the program in detail by simulating an execution.

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Outer program

$$[(corr; drive)^*]z \leq m$$

Informally:

'Whenever we *corr*ect the speed and *drive* the train, we should never pass our mark.'

Let the start state ν be
$\{v = 35, z = 34000, m = 35000, b = 1, a = 0, s = 500, \epsilon = 5, \tau = 0\}$

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corr

$$(?m - z < s; a := -b) \cup (?m - z \geq s; a := 0)$$

ν:

$\{v = 35, z = 34000, m = 35000, b = 1, a = 0, s = 500, \epsilon = 5, \tau = 0\}$

Let's compute ρ of this program, assuming ν. We need the union of the left and right sides.

Left side is $\emptyset$ because
$(?m - z < s) = (?35000 - 34000 < 500) = false.$

Right side:

$(?m - z \geq s) = (?35000 - 34000 \geq 500) = true$ gives us (ν, ν)

$a := 0$ gives us $(\nu, \nu[a \mapsto 0]) = (\nu, \nu)$

'We are not inside the safety margin, do not adjust speed by (de)accelerating.'

drive

$$\tau := 0; (\dot{z}{=}\, v, \dot{v}{=}\, a, \dot{\tau}{=}\, 1 \& v \geq 0 \wedge \tau \leq \epsilon)$$

ν:$\{v = 35, z = 34000, m = 35000, b = 1, a = 0, s = 500, \epsilon = 5, \tau = 0\}$

First, τ is set to 0, not changing ν. Then we need to calculate the values of the f function for each variable:

Let $t = 5$. $val(\omega, .) = val(f(5), .)$:

- $val(\omega, z) = z(5) = 35 * 5 + 34000 = 34175$
- $val(\omega, v) = v(5) = 0 * 5 + 35 = 35$
- $val(\omega, \tau) = \tau(5) = 1 * 5 + 0 = 5$

corr

Later...

$$(?m - z < s; a := -b) \cup (?m - z \geq s; a := 0)$$

ν:

$\{v = 35, z = 34525, m = 35000, b = 1, a = 0, s = 500, \epsilon = 5, \tau = 0\}$

The left side is not empty now because

$(?m - z < s) = (?35000 - 34525 < 500) = true.$

The ω state is now $\nu[a \mapsto -b] = \nu[a \mapsto -1]$

Dynamic logic for Hybrid systems

drive

$$\tau := 0; (\dot{z}{=}\, v, \dot{v}{=}\, a, \dot{\tau}{=}\, 1 \& v \geq 0 \wedge \tau \leq \epsilon)$$

ν:
$\{v = 35, z = 34525, m = 35000, b = 1, a = -1, s = 500, \epsilon = 5, \tau = 5\}$

First, τ is set to 0. Second, we see that the speed v now changes over time, and this has effect on the evolution of z (the train won't travel as far when it slows down).

- $val(\omega, v) = val(\omega, v(5)) = -1 * 5 + 35 = 30$
- $val(\omega, z) = val(\omega, z(5)) = \int_0^5 (-1t + 35)dt + 34525 =$
  $[-0.5t^2 + 35r]_0^5 + 34525 = 162.5 + 34525 = 34687.5$

drive

Since *corr* won't change anything (the train is braking until it stops) we will combine several instances of *drive* for brevity (time now passes for 19 seconds):

- $val(\omega, v) = val(\omega, v(19)) = -1 * 19 + 35 = 11$
- $val(\omega, z) = val(\omega, z(19)) = \int_0^{19}(-1t + 35) + 34525 = [-0.5t^2 + 35t]_0^{19} + 34525 = 484.5 + 34525 = 35009.5$

After 19 seconds the train unfortunately crosses the 'mark'.

(The train stops after 35 seconds with $z = 35137.5$)

Note!

The hybrid programs are *not* meant to be executed, but analyzed.
The previous example was just to explain how they work!

Verification calculus

Assuming a system on the form:

$$\psi \rightarrow [\alpha]\phi$$

A sequent calculus is used for verfying whether α always will satifsfy ϕ, given initial requirement ψ.

Rules of the calculus

The rules are comprised of the following:

- Rules of propositional logic
- Rules of dynamic logic
- Rules for discrete change and continuous evolution
- Rules for quantifier elimination and an induction schema

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Dynamic logic

$$D1 : \frac{\phi \wedge \psi}{<?\phi > \psi} \qquad D2 : \frac{\phi \rightarrow \psi}{[?\phi]\psi} \qquad D3 : \frac{< \alpha > \phi \vee < \gamma > \phi}{< \alpha \cup \gamma > \phi}$$

$$D4 : \frac{[\alpha]\phi \wedge [\gamma]\phi}{[\alpha \cup \gamma]\phi} \qquad D5 : \frac{\phi \vee < \alpha; \alpha^* > \phi}{< \alpha^* > \phi}$$

$$D6 : \frac{\phi \wedge [\alpha; \alpha^*]\phi}{[\alpha^*]\phi} \qquad D7 : \frac{< [\alpha] >< [\gamma] > \phi}{< [\alpha; \gamma] > \phi}$$

Discrete change

D8:

$$\frac{\phi_x^\theta}{< [x := \theta] > \phi}$$

Only applicable if the substitution introduces no new bindings

Continuous evolution

The rule D14 says that we can substitute the derivatives of functions for the functions themselves (we go from $\dot{x}= \theta$ to $x := y_x(t)$). The parameter t (time) may have many values, so we introduce an universial quantifier.

$$\frac{\forall t \geq 0(\overline{\chi} \rightarrow [x := y_x(t)]\phi)}{[\dot{x}= \theta \& \chi]}\phi$$

So what is $\overline{\chi}$? An assertion that χ is never violated by any value that t may have during the continuous evolution:
$\forall 0 < \overline{t} < t, [x := y_x(\overline{t})]\chi$

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D13 is the corresponding rule for the <> modality.

$$\frac{\exists t \geq 0(\bar{\chi} \rightarrow < x := y_x(t) > \phi)}{< \dot{x} = \theta \& \chi >} \phi$$

The tool Mathematica is used in the author's implementation for D13 and D14.

Induction schema

D14:

$$\frac{\vdash p \qquad \vdash [\alpha^*](p \rightarrow [\alpha]p)}{\vdash [\alpha^*]p}$$

Quantifier elimination

D9 (D10, D11 and D12 are similar)

$$\frac{qelim(\exists x \wedge_i (\Gamma_i \vdash \Delta_i))}{\Gamma \vdash \Delta, \exists x \phi}$$

These rules combine side deductions with quantifier elimination.

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Let's demonstrate the rule D9 (and some of the previous rules) with an example: We are interested in knowing the following: If we start an emergency brake procedure - can we expect to cross the 'mark'?

Note!

We are now interested in universal properties of the system. For example whether $z \leq m$ instead of one special case where $34000 < 35000$.

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If we start an emergency brake procedure - can we expect to cross the 'mark'?

$$\vdash v > 0 \land z < m \rightarrow <\dot{z}= v, \dot{v}= -b > z \geq m$$

We use a (well-known) rule from propositional logic before we come to the interesting stuff:

$$\frac{v > 0, z < m \vdash <\dot{z}= v, \dot{v}= -b > z \geq m}{\vdash v > 0 \land z < m \rightarrow <\dot{z}= v, \dot{v}= -b > z \geq m}$$

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The rule D13 can now be applied:

$$\frac{v > 0, z < m \vdash \exists t \geq 0 < z := -\frac{b}{2}t^2 + vt + z > z \geq m}{v > 0, z < m \vdash <\dot{z}= v, \dot{v}= -b > z \geq m}$$

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Now we need to remove the quantifier, which we do by a side deduction where we make t a free variable and try to remove it from inside the modalities.

From...

$$v > 0, z < m \vdash \exists t \geq 0 < z := -\frac{b}{2} + vt + z > z \geq m$$

...to start of side deduction...

$$v > 0, z < m \vdash t \geq 0 \wedge < z := -\frac{b}{2}t^2 + vt + z > z \geq m$$

... and, by propositional logic, get

$$\frac{v > 0, z < m \vdash t \geq 0 \qquad v > 0, z < m \vdash < z := -\frac{b}{2}t^2 + vt + z > z \geq m}{v > 0, z < m \vdash t \geq 0 \wedge < z := -\frac{b}{2}t^2 + vt + z > z \geq m}$$

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We are done with the left branch, on the right branch we can use substitution (D8):

$$\frac{v > 0, z < m \vdash -\frac{b}{2}t^2 + vt + z \geq m}{v > 0, z < m \vdash < z := -\frac{b}{2}t^2 + vt + z > z \geq m}$$

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Now we feed the conjuction of the open branches into Mathematica (or a similar tool), in order to eliminate the quantifier:

$qelim(\exists t(v > 0, z < m \vdash t \geq 0 \wedge -\frac{b}{2}t^2 + vt + z \geq m))$

Quantifier elimination - manually

$-\frac{b}{2}t^2 + vt + (z - m) \geq 0$ is a quadratic (in)equation. It has solutions only if the polynomial has roots:

$$t = \frac{-B \overset{+}{-} \sqrt{B^2 - 4AC}}{2A}, \frac{-v \overset{+}{-} \sqrt{v^2 - 4(-\frac{b}{2})(z-m)}}{2(-\frac{b}{2})} = \frac{v \overset{+}{-} \sqrt{v^2 + 2b(z-m)}}{b}$$

So, the root only exists if $v^2 + 2b(z - m) \geq 0$, that is $v^2 \geq 2b(m - z)$.

Soundness and incompleteness

All the rules of the logic are locally sound, but the logic is incomplete.