Thoralf Skolem 1887-1963

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1 Life

Thoralf Albert Skolem was born on 23 May 1887 in Sandsvær close to and later incorporated into the city of Kongsberg in Southern Norway — 80 km west of Oslo. He grew up in a rural environment. His forefathers had been farmers at the farm Skoli for generations and his father was an elementary teacher in the village.

He studied at the University of Oslo (then Kristiania) and graduated in 1913. In 1915-16 he was in Göttingen, Germany and had a position 1930-38 at a research institute in Bergen, Norway. Else he remained at the University of Oslo. He retired in 1957, but was active in research until his death on 23 March 1963.

2 Sources

The main source is *"Selected Works in Logic. Universitetsforlaget 1970"* with Jens Erik Fenstad as editor. There we have

- Jens Erik Fenstad: "Thoralf Albert Skolem in memoriam" – a biographical sketch (8 pages)
- Hao Wang: "A survey of Skolem's work in logic" – a scientific assessment of his work in logic (46 pages)
- 50 selected articles in logic (660 pages)

- The bibliography of 192 items by Skolem with indications of where the items where reviewed.

The "Selected works" has stood the test of time. The 50 selected articles contains all the important ones. In reference to Skolems articles we use the numbering from the bibliography.

In addition we have the Skolem chapter in *"Handbook of The History of Logic. Volume 5. Logic from Russell to Church. Dov Gabbay and John Woods (eds). North-Holland 2009."*. There we have

- The bibliography of Skolem.
- A reprint of Wangs assessment.
- An update of the biographical sketch.
- Supplementary notes by Jens Erik Fenstad where he refers to more recent scholarship on Skolem.

And then from the Department of Mathematics, University of Oslo. I started as a student there in the Fall of 1962 and never met Skolem. But there were many stories about him in the Department.

3 Background and style

Hao Wang writes in the Selected Works "Skolem has a tendency of treating general problems by concrete examples. Often proofs seem to be presented in the same order as he came to discover them. This results in a fresh informality as well as a certain inconclusiveness. Many of his papers give the impression of reports on work in progress. Yet his ideas are often pregnant and potentially capable of wide applications. He was very much a "free spirit": he did not belong to any school, he did not found any school of his own, he did not usually make heavy use of known results in more specialized developments, rather he was very much an innovator and most of his papers can be read and understood by people without much specialized knowledge. It seems quite likely that if he were young today, logic, in its more developed stage, would not have appealed to him."

A main change since 1970 is that there are now more ways to do logic. The influence from computer science with the emphasis on algorithms and complexity has changed much. This logic would have appealed to Skolem. But Skolems way of doing mathematics had always found a place within mathematics.

Skolems other scientific interest were in number theory (especially Diophantine equations) and combinatorics. He published 2nd edition of Netto: Lehrbuch der Combinatorik (1927c) with new chapters and on his own "Diophantische Gleichungen" (1938c).

4 Main achievements

Skolem published almost 200 papers, but his most interesting work is in his first 20 years.

- Starting lattice theory
- Skolem-Löwenheim theorem
- Skolem functions
- Method of elimination of quantifiers
- Term models are sufficient to show satisfiability
- Primitive recursive arithmetic
- Axioms of set theory
- Non-standard models of arithmetic and set theory

Skolem worked mostly on his own. He had few students — practically none in logic. He usually gave lectures in algebra and number theory. He preferred not to lecture in logic – he thought it hard to to lecture with all the syntactical details used in logic and it took too long time to get to what he thought was the interesting stuff.

5 Skolems development

Below we shall sketch a path through his most important work. The work was mostly developed on his own. There are few references to results by other, but his work was read and absorbed in the logical tradition.

5.1

Skolem worked in what van Heijenoort called the algebraic tradition of logic (Boole – Peirce – Schröder). He used Schröders notation in his papers through the 1930's.

5.2

In his first works – 1913b and 1919a – he introduced what much later was called lattice theory and especially distributive lattices. He had to remind others in 1936g about this earlier work.

Formulas were used to express propositions, and a typical problem was whether a formula could be satisfied. This is simple in propositional logic, but complicated when we come to quantifiers – it corresponds to the Entscheidungsproblem which we know is unsolvable. Skolems main work concerned the treatment of quantifiers – if satisfiability of a formula is like solving an equation, then how should quantifiers be treated?

5.3

In Diophantine equations we can — using new variables — show that it suffices to consider system of equations of degree 2. For say we start with

$$x^3 + y^2 z = 0$$

then introduce new variables r, s, t and the system of equations

$$\begin{aligned} x^2 &= r \\ rx &= s \\ y^2 &= t \\ s + tz &= 0 \end{aligned}$$

The first system is solvable if and only if the second is.

In analogy with this Skolem did a similar thing in logic. Let us say that we start with the formula

$$\forall x.\exists y.\forall z.Rxyz$$

and then replace it with a conjunction of (universally quantified) equivalences with new predicates S, T, and U.

$$\begin{array}{rcl} \forall z.Rxyz & \leftrightarrow & Sxy \\ \exists u.Sxu & \leftrightarrow & Tx \\ \forall v.Tv & \leftrightarrow & U \end{array}$$

and replacing biconditionals with conditionals and changing names for some bounded variables

$$\begin{array}{rcl} \forall z.Rxyz & \leftarrow & Sxy \\ \forall a.Rxya & \rightarrow & Sxy \\ \exists u.Sxu & \leftarrow & Tx \\ \exists b.Sxb & \rightarrow & Tx \\ \forall v.Tv & \leftarrow & U \\ \forall c.Tc & \rightarrow & U \end{array}$$

Writing this system out in prenex form we can get the quantifiers with $\forall$-quantifiers outside the $\exists$-quantifiers starting with an outermost prenex

$$\forall x \forall y \forall z \forall b \forall v \exists a \exists u \exists c$$

and a quantifierfree matrix inside

$$Mxyzuvabc$$

We say that we have a formula of type $\forall^*\exists^*$ and the reduction of satisfiability to such formulas.

Here is Skolems first result about quantifiers. For solving logical formulas it is sufficient to consider formulas of form $\forall^*\exists^*$ provided we are allowed to introduce new predicates.

5.4

Skolem learned about Löwenheims theorem in Göttingen 1915-16. He thought the proof of it had gaps and developed his own proof. The formulas $\forall^*\exists^*$

can be interpreted as giving a functional connection. In the example above we have the functional connections going from any *xyzbv* to selecting appropriate *auc*. And we have the Skolem-Löwenheim theorem — if a formula is satisfiable, then it is satisfiable in a countable domain. Skolem proved this in 1920c using axiom of choice to realize the functional connections. The $\forall^*\exists^*$-formulas are straightforward to interpret.

5.5

In 1922c Skolem has sharpened this result. He gave a process to find out whether a first order formula is satisfiable

- We start with a formula in $\forall^*\exists^*$-form.
- We construct a possible model by stages and start with the possible models of the matrix of the formula where we have substituted in constants from the formula.
- At each stage we introduce more constants from the $\forall^*\exists^*$-connection and look at the models extending the models constructed so far.
- Each stage is performed in a constructive way and we construct a tree where at each node we have a model. The model at a node extends the models at the nodes below it.
- It may happen that at some stage we get no models. Then the original formula would be refutable.
- Or it may happen that we get an infinite branch in the tree and this gives a model of the formula.

In this way Skolem uses a variant of the termmodel — built up from Skolem functions — instead of the axiom of choice. For the countable case this works fine and gives not only the Skolem-Löwenheim result but also the completeness of first order logic. In 1928a coming back to the result he showed that the procedure is complete.

Why did not Skolem state and explicitly prove the completeness theorem? His interests seem to lie elsewhere. He was interested in procedures for finding whether formulas are satisfiable. This is quite clear in 1928a.

There he showed the completeness of a procedure for checking whether a formula F in $\forall^*\exists^*$-form is satisfiable. The formula gives functional connection which hopefully can be realized. We get a finitary process where we build up the model.

These are all the ingredients to get the completeness theorem of Gödel from 1930. But Skolem never formulated the completeness theorem as a result. It is noteworthy that in his later lectures he never mentioned the completeness theorem as an interesting result.

5.6

In 1922b Skolem introduced the modern axiom system for set theory. Zermelo with his Aussonderungsaxiom and Fraenkel with his replacement axiom used the unanalyzed notion of definite proposition. Skolem used axiom schemas with ordinary first order formulas – as we now do. Hao Wang mentions that because of this one should perhaps use Skolems name in the name of the axioms of set theory.

Skolem was not interested in set theory as a foundational theory. He observed that if set theory (in his formulation) had a model, then it had a countable model. In his view it would not work as a foundational theory.

The unintended models of set theory leads up to other unintended models. Skolem mentions in 1922b that one could have models of number theory with unintended versions of induction.

5.7

Skolem had in Göttingen learnt about Russell and Whiteheads "Principia Mathematicae" and disagreed with their foundational view. In 1923a he published a remarkable work with an alternative foundation. In modern terms he introduced

Datastructure: Natural numbers as built from 1 and the successor

Programming language: The primitive recursive functions

Programming logic: Primitive recursive arithmetic

Skolem showed how to develop number theory up to prime number decomposition within this framework. Both Grassman (1860) and Dedekind (1890) had used the primitive recursive definition of addition and multiplication, but Skolem was the first to use it towards expressing a substantial field of knowledge. Instead of quantifiers Skolem showed how to use free variable reasoning. He used

- defining equations of primitive recursive functions
- induction over quantifier free formulas which may contain free variables

In the development of number theory Skolem did not use the whole of primitive recursive functions. He used especially bounded sums and products — what is later called the Kalmar elementary functions.

In 1927-28 Hilberts students Sudan and Ackermann gave examples of computable functions which were not primitive recursive. Skolem observed that the graph of the functions are still primitive recursive and continued to look for good examples from mathematics where we had to go beyond the primitive recursive — without finding any. An attempt is in 1956c.

5.8

The quantifiers are stumbling blocks in going from propositional logic to first order logic. Skolem showed that in a number of cases he could eliminate them

- Skolem introduced the method of elimination of quantifiers in 1919b. He showed there how it could be done with a special sort of Boolean algebra — the subsets of a given set.
- In 1929a he treated dense linear orders and gave a new proof that the theory is decidable.
- In 1930c he used quantifier elimination on Presburger arithmetic — arithmetic with only addition. Skolem treated also the theory of arithmetic with only multiplication, and showed that it is decidable.

The elimination of quantifiers are often very sensitive to the language used, and requires ingenuity in getting the right formulations.

5.9

From the Skolem-Löwenheim theorem we get unintended models of set theory. Skolem remarked in 1922a that we also get unintended models of arithmetic where induction is problematic. In 1933d and 1934b he gave a direct construction of an unintended model of arithmetic. His construction is a version of the ultrapower construction done 30 years later.

5.10

It is clear that Skolem was quite close to the incompleteness theorem of Gödel. He had given unintended models of arithmetic and also shown how to develop much of syntax within number theory. When he later was asked about Gödels contribution he emphasized the Gödels β-function — constructed using the Chinese remainder theorem. With the β-function Gödel was able to treat finite sequences of information within arithmetic with addition and multiplication. This step was lacking in Skolems development.

If Skolem had chosen a better datastructure in 1923a than the natural numbers, he could have come much closer. In the datastructure of binary trees we can explicitly define a β-function using bounded quantifiers and bounded search. (This works also with the datastructure of binary words and the datastructure of hereditarily finite sets.) The problems comes with the natural numbers as a datastructure. There we use addition and multiplication as functions and get the β-function defined using an extra unbounded quantifier. This is sufficient for the incompleteness theorem. Skolem showed in 1930c that we could not work with neither addition alone nor with multiplication alone.

5.11

An interesting application of coding syntax was done in 1958d. We may have finite axiomsets and sets where we use axiom schemas. Skolem showed that any theory formulated with axiom schemas, could also be formulated with a finite axiom set. This is similar to set theory where we have axiom schemas in Zermelo-Fraenkel and finite sets of axioms in Gödel-Bernays axiom system.

Say we have a system $\mathcal{S}$ formulated with axiom schemas. We then consider a new system $\mathcal{S}*$ where we have

- two sorts of individuals
  - finite sequences of individuals from $\mathcal{S}$
  - predicates from $\mathcal{S}$
- predicates between the sorts expressing
  - identity between sequences
  - the finite sequences are of same length
  - concatenation of finite sequences
  - a finite sequence satisfies a predicate

In $\mathcal{S}*$ we can give a finite set of axioms expressing all the axioms of $\mathcal{S}$ including the axiom schemas. The development is fairly straightforward, but is also a typical Skolem argument. We refer the reader to the paper, or let the reader reconstruct the Skolem argument by showing that in $\mathcal{S}*$ we can express the following

- that a sequence of individuals consists of only one element
- the atomic formulas of $\mathcal{S}$
- the permutations of sequences of individuals
- propositional operations on formulas from $\mathcal{S}$
- quantifying formulas from $\mathcal{S}$
- axiom schemes from $\mathcal{S}$

We then simulate $\mathcal{S}$ in $\mathcal{S}*$ using finitely many axioms.

5.12

Skolem continued his work throughout his life. There are papers on

- Reduction classes of satisfiability
- Recursive function theory
- Foundations of set theory
- and much more

But his main contributions to logic are the items mentioned above.

6 References

For a full bibliography see his Selected Works.

1913b: *Om konstitusjonen av den identiske kalkyls grupper. (On the structure of groups in the identity calculus.)*
1919a: *Untersuchungen über die Axiome des Klassenkalküls und über Produktations- und Summationsprobleme, welche gewisse Klassen von Aussagen betreffen.*
1920a: *Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit und Beweisbarkeit mathematischen Sätze nebst einem Theoreme über dichte Mengen.*
1922b: *Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre.*
1923a: *Begründung der elementären Arithmetik durch die rekurrierende Denkweise ohne Anwendung scheinbarer Veränderlichen mit unendlichem Ausdehnungsbereich.*
1928a: *Über die mathematische Logik.*
1929a: *Über die Grundlagendisskussionen in der Mathematik.*
1930c: *Über einige Satzfunktionen in der Arithmetik.*
1931d: *Über die symmetrisch allgemeinen Lösungen im identischen Kalkul.*
1933d: *Über die Unmöglichkeit einer Charakterisierung der Zahlenreihe mittels eines endlichen Axiomensystems.*
1936g: *Über gewisse "Verbände" oder "Lattices".*
1956c: *An ordered set of arithmetic functions representing the least ε-number.*
1958d: *Reduction of axiom systems with axiom schemes to systems with only simple axioms.*