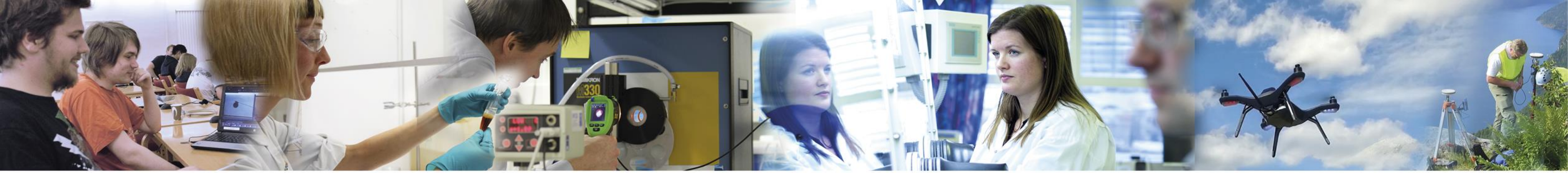

UiO : Department of Technology Systems
University of Oslo

TEK5010/9010 - Multiagent systems 2021
Lecture 9

Voting – making group decisions

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Highlights lecture 9 – Voting*

- How do we aggregate individual preferences into social choice?
- Different voting procedures
- Arrow’s theorem and desirable properties
- Strategic manipulation and Gibbard-Satterthwaite theorem

*Wooldridge, 2009: chapter 12

Making group decisions

- Social choice theory, or voting theory, is about making group decisions (as opposed to individual decisions).
- Group decisions can be viewed as games because agents will take into account their own preferences as well as those of others in order to bring about their most preferred outcome.
- This leads to tactical voting and strategic manipulation of elections (or choice of group strategy).

Social choice theory

Basic setting:

$$Ag = \{1,2,\dots,N\}$$

where Ag is the set of agents or voters in group

$|Ag| = N$ is number of agents

N is assumed to be finite and odd (to break ties)

Social choice theory

The set of possible outcomes (or candidates)

$$\Omega = \{\omega_1, \omega_2, \ldots\}$$

where ω_i is a possible outcome

$|\Omega| = k$ is number of possible outcomes

$|\Omega| = 2$ is pairwise election

$|\Omega| > 2$ is general election

Social choice theory

Preference ordering of agents:

$$\boldsymbol{\varpi}_i = (\omega_3, \omega_1, \dots, \omega_k)$$

where $\boldsymbol{\varpi}_i$ is preference ordering (based on rank) of agent i

ω_k is one out of k possible outcomes in Ω

and

$\omega \succ_i \omega'$ means ω is preferred over ω' for agent i in $\boldsymbol{\varpi}_i$

Social choice theory

Preference aggregation is the fundamental problem in social choice theory:

How do we combine the different agents' preference ordering in order to derive a group decision?

Or more specific, how do we generate a social preference order over possible outcomes?

Social choice theory

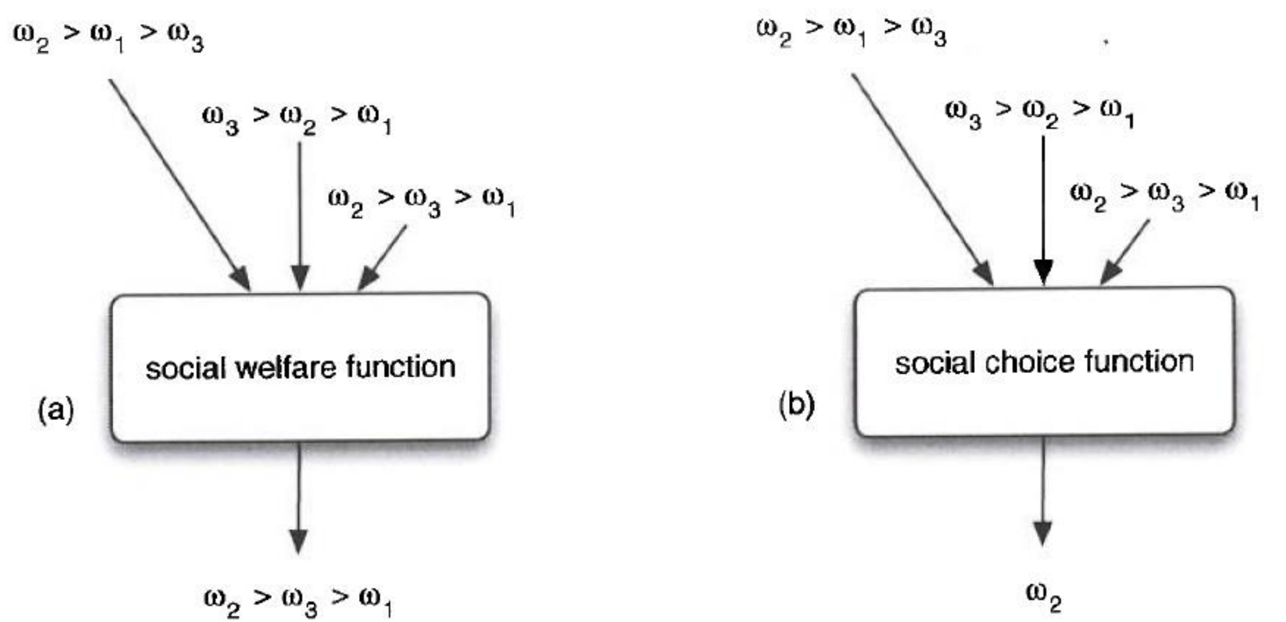


Image: Figure 12.1, Wooldridge 2009

Social preference order

Social welfare function:

$$f : \Pi(\Omega) \times \cdots \times \Pi(\Omega) \to \Pi(\Omega)$$

where $\Pi(\Omega)$ is a preference ordering over outcomes

f is a ranking producing a social preference order

and

$\omega \succ^* \omega'$ means ω ranked over ω' in the social outcome

Social preference order

Social choice function:

$$f : \Pi(\Omega) \times \cdots \times \Pi(\Omega) \rightarrow \Omega$$

where $\Pi(\Omega)$ is a preference ordering over outcomes

f is one of the outcomes in Ω

Voting procedures

- Plurality
- Simple majority voting
- Sequential majority voting
- Borda count
- Slater ranking
- Dictatorship
- Second-order Copeland

…

Plurality

Every voter submits their preference order and the winner is the outcome ranked first most times.

This is the simplest and best known voting procedure, e.g. marking a candidate on a ballot.

Plurality is straightforward to implement and easy to understand by the voters.

Plurality

However,

- Plurality is vulnerable to strategic manipulation and tactical voting
- and it reveals Condorcet’s paradox

Condorcet's paradox

Assume three voters $Ag = \{1, 2, 3\}$ having three possible outcomes $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with preferences as follows:

$$\boldsymbol{\varpi}_1 = (\omega_1, \omega_2, \omega_3) \Leftrightarrow \omega_1 \succ_1 \omega_2 \succ_1 \omega_3$$
$$\boldsymbol{\varpi}_2 = (\omega_3, \omega_1, \omega_2) \Leftrightarrow \omega_3 \succ_2 \omega_1 \succ_2 \omega_2$$
$$\boldsymbol{\varpi}_3 = (\omega_2, \omega_3, \omega_1) \Leftrightarrow \omega_2 \succ_3 \omega_3 \succ_3 \omega_1$$

Selcting $\omega_1$means 2/3 of the voters would rather prefer ω_3, and same for goes for selecting ω_2 or ω_3.

Condorcet's paradox

There are scenarios in which no matter which outcome we choose, a majority of voters will be unhappy with the outcome.

It all depends on the preference ordering of the voters.

Tactical voting

Is it possible for voters to produce a better outcome in the example by manipulating the representation of the their preferences?

$$\boldsymbol{\varpi}_1 = (\omega_1, \omega_2, \omega_3) \Leftrightarrow \omega_1 \succ_1 \omega_2 \succ_1 \omega_3$$
$$\boldsymbol{\varpi}_2 = (\omega_3, \omega_1, \omega_2) \Leftrightarrow \omega_3 \succ_2 \omega_1 \succ_2 \omega_2$$
$$\boldsymbol{\varpi}_3 = (\omega_2, \omega_3, \omega_1) \Leftrightarrow \omega_2 \succ_3 \omega_3 \succ_3 \omega_1$$

Tactical voting

Voter 3 can for instance manipulate the representation of its preferences by untruthfully say that $\omega_2 \succ_3 \omega_3 \rightarrow \omega_3 \succ_3 \omega_2$.

$$\boldsymbol{\varpi}_1 = (\omega_1, \omega_2, \omega_3) \Leftrightarrow \omega_1 \succ_1 \omega_2 \succ_1 \omega_3$$
$$\boldsymbol{\varpi}_2 = (\omega_3, \omega_1, \omega_2) \Leftrightarrow \omega_3 \succ_2 \omega_1 \succ_2 \omega_2$$
$$\boldsymbol{\varpi}_3' = (\omega_3, \omega_2, \omega_1) \Leftrightarrow \omega_3 \succ_3 \omega_2 \succ_3 \omega_1$$

$$(\boldsymbol{\varpi}_3 = (\omega_2, \omega_3, \omega_1) \Leftrightarrow \omega_2 \succ_3 \omega_3 \succ_3 \omega_1)$$

Now, ω_3 is a more preferred choice than $\omega_1$for voter 3 and 2!

Simple majority voting

Simple majority voting is plurality with only two possible outcomes $|\Omega| = 2$.

Not so easy to manipulate…

But in reality there are often more than 2 possible outcomes.

Sequential majority election

A series of plurality elections in order to determine a winner.

Typically, a selection of pairwise elections (i.e. simple majority voting) based on

1. linear sequential elections or,
2. balanced binary tree election.

Sequential majority election

Linear sequential election:

Balanced binary tree election:

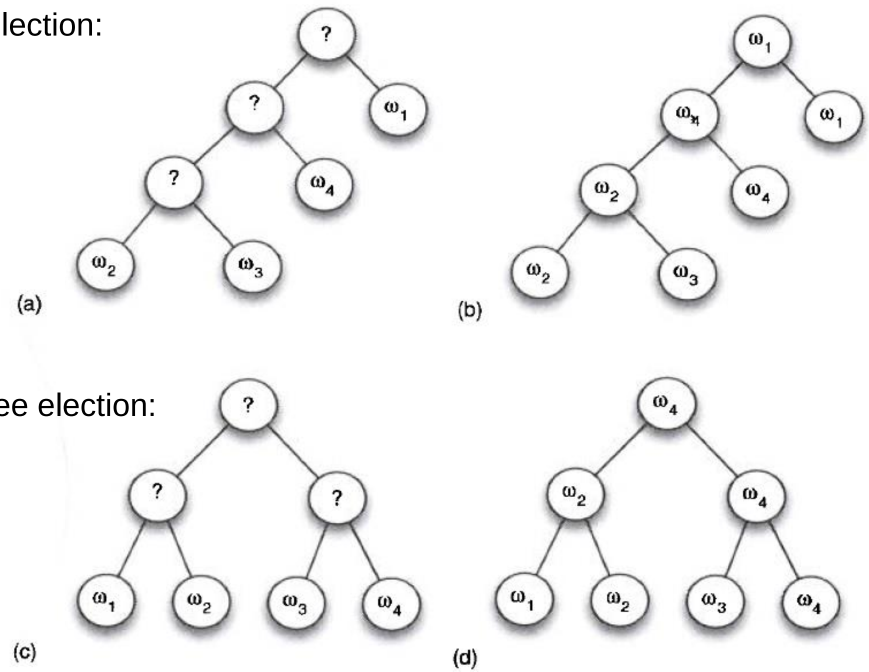


Image: Figure 12.2, Wooldridge 2009

Sequential majority election

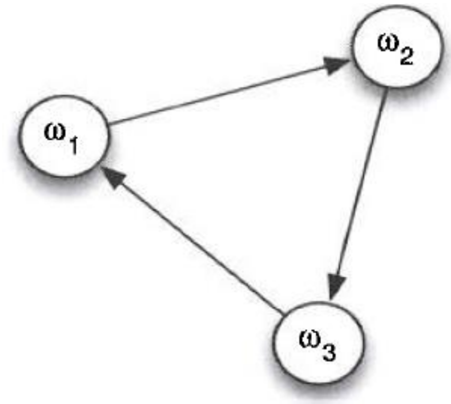
Election agenda:

- The order of arranging sequential elections

  $\omega_1, \omega_2, \omega_3$ meaning ω_1 vs. ω_2, and then ω_1/ω_2 vs. ω_3
  $\omega_2, \omega_3, \omega_1$ meaning ω_2 vs. ω_3, and then ω_2/ω_3 vs. ω_1

- The outcome is generally sensitive to order, either by random selection or manipulated by electioneer.

Majority graph



A directed graph constructed from voter preferences.

1. Nodes correspond to outcomes in Ω
2. Edges correspond to majority outcome between pairwise node elections

Image: Figure 12.3, Wooldridge 2009

Majority graph

Properties:

1. Completeness
   For any two outcomes ω_i and ω_j, we must have either ω_i defeat ω_j or ω_j defeat ω_i
2. Asymmetry
   If ω_i defeats ω_j then ω_j cannot defeat ω_i
3. Irreflexivity
   ω_i will never defeat itself

Majority graph

An outcome is a *possible winner* if there are some election agenda that results in that outcome to be the overall winner in a sequential majority election.

In a majority graph this is checked by evaluting the connectedness between nodes, i.e. is there a path from node ω_i to any other node ω_j ('the graph reachability problem')?

Condorcet winner

The Condorcet winner is the outcome that is the overall winner for all possible election agendas.

In a majority graph the Condorcet winner is the node that is connected from the node to every other node.

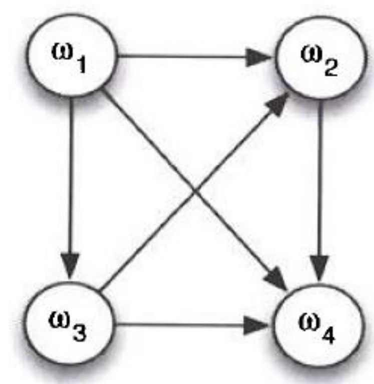


Image: Figure 12.3, Wooldridge 2009

Dictatorship

A social welfare function is said to be a dictatorship if

$$f(\boldsymbol{\varpi}_1, \ldots, \boldsymbol{\varpi}_i, \ldots \boldsymbol{\varpi}_N) = \boldsymbol{\varpi}_i$$

where $\boldsymbol{\varpi}_i$ is the preference order of voter i

Consequently, in dictatorship, the social outcome is only dependent on voter i.

Borda count

The Borda count (BC) for outcome ω_j is given by:

$$BC_{\omega_j} = \sum_{i=1}^{N} k - rank\left(\boldsymbol{\varpi}_i(\omega_j)\right)$$

where $|\Omega| = k$ is the number of possible outcomes

N is number of voters

$\boldsymbol{\varpi}_i$ is social preference order of voter i

Borda count

Borda count take into account other information in the voter preference list than top rank.

The social outcome is the outcome with maximal Borda count.

Slater ranking

Used for breaking cycles in majority graphs. The slater rank for a social ordering is how many edges must be ‘flipped’ in the cyclic majority graph to produce that particular social order.

The slater rule is to choose the social ordering that minimizes the disagreement between the majority graph and the social choice, i.e. the order with lowest slater rank number.

Computing the Slater rank is NP-hard.

Slater ranking

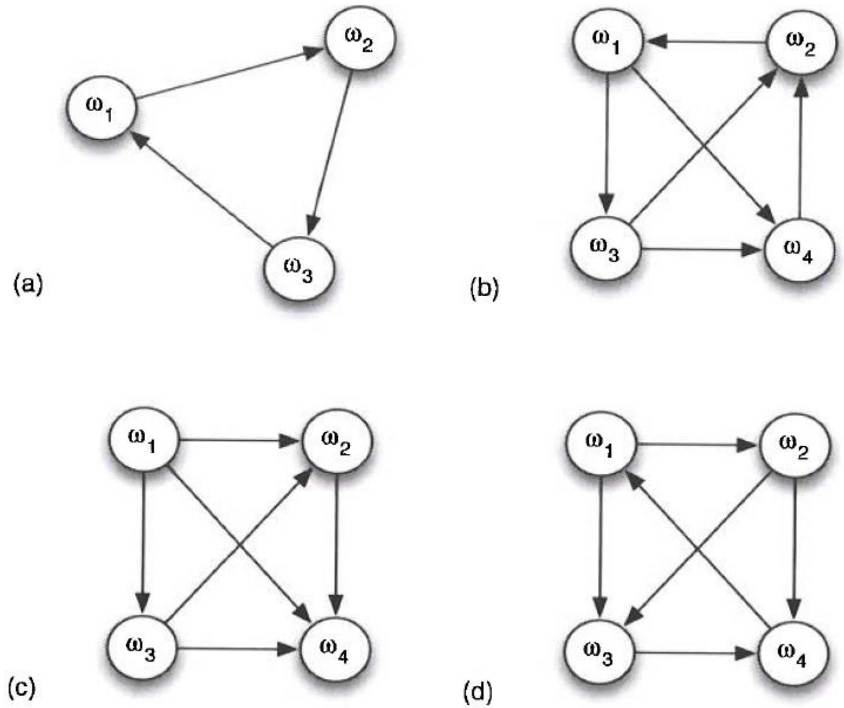


Image: Figure 12.3, Wooldridge 2009

Slater ranking

Examples of slater count of different social choice ordering:

1. $\omega_1 \succ^* \omega_2 \succ^* \omega_3 \succ^* \omega_4$
2. $\omega_1 \succ^* \omega_2 \succ^* \omega_4 \succ^* \omega_3$

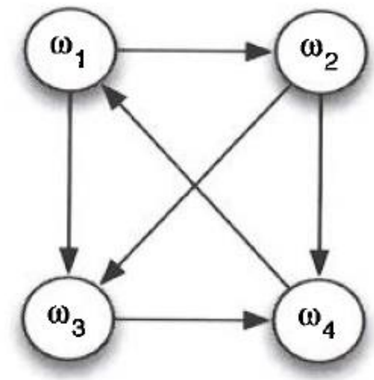


Image: Figure 12.3, Wooldridge 2009

Arrow’s theorem

Assuming voters have 3 or more distinct alternatives, there exist *no* ranked voting electoral system that can convert the ranked preferences of individuals into as social preference order while at the same time also meet a set of specific ‘desirable’ criteria:

1. Unrestricted domain
2. Pareto efficiency
3. Independence of irrelevant alternatives (IIA)
4. Non-dictatorship

Unrestricted domain

The unrestricted domain condition states that all preferences of all voters are allowed,

meaning that the preference order of the voters should be complete and that the social preference order should be deterministic.

Pareto efficiency

The Pareto condition states that there is no other outcome that makes one voter better off without making any other voter worse off, i.e. if all voters $\omega \succ \omega' \Rightarrow \omega \succ^* \omega'$.

This condition i satisfied for Plurality, Borda and dictatorship but not for sequential majority election.

Independence of irrelevant alternatives (IIA)

The social preference between outcome $\omega \succ^* \omega'$ depends only on the individual preferences between $\omega \succ \omega'$.

Meaning that the ranking of all the other outcomes, not changing the relative ranking of individual ranking of $\omega \succ \omega'$, should not affect the social ranking of $\omega \succ^* \omega'$.

Dictatorship satisfies this criterion, but Plurality, Borda and sequential majority election do not.

Non-dictatorship

If the non-dictatorship condition is dropped as a criterion then the dictatorship satisfies Arrow’s theorem!

Strategic manipulation

A social welfare function is manipulable if there exist $\boldsymbol{\varpi}_i{}'$

$$f(\boldsymbol{\varpi}_1, \ldots, \boldsymbol{\varpi}_i{}', \ldots \boldsymbol{\varpi}_N) \succ_i f(\boldsymbol{\varpi}_1, \ldots, \boldsymbol{\varpi}_i, \ldots \boldsymbol{\varpi}_N)$$

where $\boldsymbol{\varpi}_i$ is the preference order of voter i

Meaning that the social outcome could be improved for some voter i by unilaterally misrepresenting i’s preference order.

Gibbard-Satterthwaite theorem

Assuming voters have 3 or more distinct alternatives, according to the GS theorem, in general there exist *no* voting protocol, except for dictatorship, that is non-manipulable.

However,

1. strategic manipulation might be hard to compute, e.g. Second-order Copeland.
2. also, uncertainties could make manipulation strategies more difficult to obtain.

Highlights lecture 9 – Voting*

- How do we aggregate individual preferences into social choice?
- Different voting procedures:
  - Plurality (Condorcet's paradox)
  - Sequential majority election (majority graph, Condorcet winner)
  - Borda count
  - Slater rule
- Arrow's theorem and desirable properties
- Strategic manipulation and Gibbard-Satterthwaite theorem

*Wooldridge, 2009: chapter 12