Lecture 3.3
Robust estimation with RANSAC

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Motivation

• Two images, captured by perspective cameras, of the same planar scene are related by a homography $H$

• This homography can be estimated from 4 or more point-correspondences between the images

• Point-correspondences can be established automatically
  – Find key points in both images
  – Represent key points by a descriptor (a vector of parameters)
  – Establish point correspondences by comparing descriptors

• The resulting set of point pairs typically contains several wrong correspondences

• A robust estimation method provides a good estimate of $H$ despite the presence of erroneous correspondences (outliers)
RANdom SAmple Consensus - RANSAC

- RANSAC is an iterative method for estimating the parameters of a mathematical model from a set of observed data containing outliers.

Mathematical model with parameters \( \alpha = (\alpha_1, \ldots, \alpha_n) \)

\[
\begin{align*}
\mathbf{x}_i, \mathbf{y}_i & \quad \text{(Observed data)} \\
\mathbf{y} & = f(\mathbf{x}; \alpha) \\
\end{align*}
\]
RANdom SAmple Consensus - RANSAC

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\[ y = f(x; \alpha) \]

Mathematical model with parameters \( \alpha = (\alpha_1, ..., \alpha_n) \)

\[ y = ax + b \]

\[ ax + by + c = 0 \]

\[ y = ax^2 + bx + c \]

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

Observed data

\((x_i, y_i)\)
• RANSAC is an iterative method for estimating the parameters of a mathematical model from a set of observed data containing outliers
  – Robust method (handles up to 50% outliers)

\[ y = f(x; \alpha) \]

Mathematical model with parameters \( \alpha = (\alpha_1, ..., \alpha_n) \)
**RANdom SAmple Consensus - RANSAC**

- RANSAC is an iterative method for estimating the parameters of a mathematical model from a set of observed data containing outliers
  - Robust method (handles up to 50% outliers)
  - The estimated model is random but reasonable

\[
S = \{(x_i, y_i)\}
\]

\[
y = f(x; \alpha)
\]

Mathematical model with parameters \( \alpha = (\alpha_1, \ldots, \alpha_n) \)
RANdom SAmple Consensus - RANSAC

- RANSAC is an iterative method for estimating the parameters of a mathematical model from a set of observed data containing outliers
  - Robust method (handles up to 50% outliers)
  - The estimated model is random but reasonable
  - The estimation process divides the observed data into inliers and outliers
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- RANSAC is an iterative method for estimating the parameters of a mathematical model from a set of observed data containing outliers
  - Robust method (handles up to 50% outliers)
  - The estimated model is random but reasonable
  - The estimation process divides the observed data into inliers and outliers
  - Usually an improved estimate of the model is determined based on the inliers using a less robust estimation method, e.g. least squares
Basic RANSAC

Objective
To robustly fit a model $y = f(x; \alpha)$ to a data set $S$ containing outliers

Algorithm
1. Estimate the model parameters $\alpha_{tst}$ from a randomly sampled subset of $n$ data points from $S$

2. Determine the set of inliers $S_{tst} \subseteq S$ to be the data points within a distance $t$ of the model

3. If this set of inliers is the largest so far, let $S_{IN} = S_{tst}$ and let $\alpha = \alpha_{tst}$

4. If $|S_{IN}| < T$, where $T$ is some threshold value, repeat steps 1-3, otherwise stop

5. After $N$ trials, stop
Basic RANSAC

Comments

• Typically the number of random samples, \( n \), is the smallest number of data points required to estimate the model.

• Assuming Gaussian noise in the data, the threshold value \( t \) should be in the region of \( 2\sigma \) were \( \sigma \) is the expected noise in the data set.

• The threshold value \( T \) is set large enough to return a satisfactory inlier set, or simply omitted.

• The maximal number of tests, \( N \), can be chosen according to how certain we want to be of sampling at least one \( n \)-tuple with no outliers.

If \( p \) is the desired probability of sampling at least one \( n \)tuple with no outliers and \( \omega \) is the probability of a random data point to be an inlier, then

\[
N = \frac{\log(1 - p)}{\log(1 - \omega^n)}
\]

\( p = 0.99 \) is standard.
**Basic RANSAC**

**Comments**

- \( N = \frac{\log(1-p)}{\log(1-\omega^n)} \) with \( p = 0.99 \)

- Typically we do not know the ratio of outliers in our data set, hence we do not know the probability \( \omega \) or the number \( N \)

- Instead of operating with a larger than necessary \( N \) we can modify RANAC to adaptively estimate \( N \) as we perform the iterations

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Adaptive RANSAC

Objective
To robustly fit a model $y = f(x; \alpha)$ to a data set $S$ containing outliers

Algorithm
1. Let $N = \infty$, $S_{IN} = \emptyset$ and #iterations = 0

2. while $N > $ iterations repeat 3-5

3. Estimate parameters $\alpha_{tst}$ from a random $n$-tuple from $S$

4. Determine inlier set $S_{tst}$, i.e. data points within a distance $t$ of the model $y = f(x; \alpha_{tst})$

5. If $|S_{tst}| > |S_{IN}|$, set $S_{IN} = S_{tst}$, $\alpha = \alpha_{tst}$, $\omega = \frac{|S_{IN}|}{|S|}$ and $N = \frac{\log(1-p)}{\log(1-\omega^n)}$ with $p = 0.99$
   Increase #iterations by 1
Example

- Fit a circle \((x - x_0)^2 + (y - y_0)^2 = r^2\) to these data points by estimating the 3 parameters \(x_0\), \(y_0\) and \(r\).
Example

- Fit a circle \((x - x_0)^2 + (y - y_0)^2 = r^2\) to these data points by estimating the 3 parameters \(x_0\), \(y_0\) and \(r\)

- The data consists of some points on a circle with Gaussian noise and some random points
Example

- **Least-squares approach**
  Separate observables from parameters:

\[
(x - x_0)^2 + (y - y_0)^2 = r^2
\]
\[
x^2 - 2xx_0 + x_0^2 + y^2 - 2yy_0 + y_0^2 = r^2
\]
\[
2xx_0 + 2yy_0 + r^2 - x_0^2 - y_0^2 = x^2 + y^2
\]

\[
\begin{bmatrix}
  x & y & 1
end{bmatrix}
\begin{bmatrix}
  2x_0 \\
  2y_0 \\
  r^2 - x_0^2 - y_0^2
end{bmatrix}
= \begin{bmatrix}
  x^2 + y^2
end{bmatrix}
\]

\[
\begin{bmatrix}
  x & y & 1
end{bmatrix}
\begin{bmatrix}
  p_1 \\
  p_2 \\
  p_3
end{bmatrix}
= \begin{bmatrix}
  x^2 + y^2
end{bmatrix}
\]

- So for each observation \((x_i, y_i)\) we get one equation

\[
\begin{bmatrix}
  x_i & y_i & 1
end{bmatrix}
\begin{bmatrix}
  p_1 \\
  p_2 \\
  p_3
end{bmatrix}
= \begin{bmatrix}
  x_i^2 + y_i^2
end{bmatrix}
\]

- From all our \(N\) observations we get a system of linear equations

\[
\begin{bmatrix}
  x_1 & y_1 & 1 & | & p_1 & | & x_1^2 + y_1^2 \\
  x_2 & y_2 & 1 & | & p_2 & | & x_2^2 + y_2^2 \\
  \vdots & \vdots & \vdots & | & \vdots & | & \vdots \\
  x_N & y_N & 1 & | & p_3 & | & x_N^2 + y_N^2
end{bmatrix}
A \begin{bmatrix}
  p_1 \\
  p_2 \\
  p_3
end{bmatrix}
= \begin{bmatrix}
  x_1^2 + y_1^2 \\
  x_2^2 + y_2^2 \\
  \vdots \\
  x_N^2 + y_N^2
end{bmatrix}
\]
Example

- We can solve this using the pseudo inverse \( \mathbf{p} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b} \)
  - This is the solution that minimize \( \| \mathbf{A} \mathbf{p} - \mathbf{b} \| \)
Example

• We can solve this using the pseudo inverse \( p = (A^T A)^{-1} A^T b \)
  – This is the solution that minimize \( \| A p - b \| \)

• NOT GOOD! All points are treated equally, so the random points shifts the estimated circle away from the desired solution
Example

• To estimate the circle using RANSAC, we need two things
  1. A way to estimate a circle from n-points, where n is as small as possible
  2. A way to determine which of the points are inliers for an estimated circle
Example

- To estimate the circle using RANSAC, we need two things
  1. A way to estimate a circle from \( n \)-points, where \( n \) is as small as possible
  2. A way to determine which of the points are inliers for an estimated circle

- The smallest number of points required to determine a circle is 3, i.e. \( n = 3 \), and the algorithm for computing the circle is quite simple
Example

- To estimate the circle using RANSAC, we need two things
  1. A way to estimate a circle from n-points, where n is as small as possible
  2. A way to determine which of the points are inliers for an estimated circle

- The distance from a point \((x_i, y_i)\) to a circle \((x - x_0)^2 + (y - y_0)^2 = r^2\) is given by
  \[
  \sqrt{(x_i - x_0)^2 + (y_i - y_0)^2} - r
  \]
Example

- To estimate the circle using RANSAC, we need two things
  1. A way to estimate a circle from n-points, where n is as small as possible
  2. A way to determine which of the points are inliers for an estimated circle

- The distance from a point \((x_i, y_i)\) to a circle \((x - x_0)^2 + (y - y_0)^2 = r^2\) is given by
  \[
  \sqrt{(x_i - x_0)^2 + (y_i - y_0)^2 - r}
  \]

- So for a threshold value \(t\), we say that \((x_i, y_i)\) is an inlier if
  \[
  \sqrt{(x_i - x_0)^2 + (y_i - y_0)^2 - r} < t
  \]
Example

Objective
To robustly fit the model \((x - x_0)^2 + (y - y_0)^2 = r^2\) to our data set \(S = \{(x_i, y_i)\}\)

Algorithm
1. Let \(N = \infty, S_{IN} = \emptyset, p = 0.99, t = 2 \cdot \text{expected noise}\) and \#iterations = 0

2. while \(N > \#iterations\) repeat 3-5

3. Estimate parameters \((x_{tst}, y_{tst}, r_{tst})\) from three random points from \(S\)

4. Determine inlier set \(S_{tst} = \{(x_i, y_i) \in S \text{ such that } \sqrt{(x_i - x_0)^2 + (y_i - y_0)^2 - r} < t\}\)

5. If \(|S_{tst}| > |S_{IN}|\), set \(S_{IN} = S_{tst}, (x_0, y_0, r) = (x_{tst}, y_{tst}, r_{tst}), \omega = \frac{|S_{IN}|}{|S|}\) and \(N = \frac{\log(1-p)}{\log(1-\omega^n)}\)

Increase \#iterations by 1
The RANSAC algorithm evaluates many different circles and returns the circle with the largest inlier set.
Example

- The RANSAC algorithm evaluates many different circles and returns the circle with the largest inlier set.

- Inliers can be used to get an improved estimate of the circle.
Robust estimation

- RANSAC is not perfect...

- Several other robust estimation methods exist
  - Least Median Squares (LMS)
  - Preemptive RANSAC
  - PROgressive Sample and Consensus (PROSAC)
  - M-estimator Sample and Consensus (MSAC)
  - Maximum Likelihood Estimation Sample and Consensus (MLESAC)
  - Randomized RANSAC (R-RANSAC)
  - KALMANSAC
  - +++
Summary

• RANSAC
  – A robust iterative method for estimating the parameters of a mathematical model from a set of observed data containing outliers
  – Separates the observed data into “inliers” and “outliers”
  – Very useful if we want to use better, but less robust, estimation methods
  – Not perfect

• Additional reading
  – Szeliski: 6.1.4

• Homework?
  – Implement a RANSAC algorithm for estimating a line