TWO-DIMENSIONAL METAMATERIAL

We have already seen how the concept of a metamaterial yields an analytic description of a planar periodic layer structure. Now let us apply the metamaterial concept to a doubly periodic array of cylinders. Let the relative permittivity be \( \varepsilon_1 \) inside the cylinders and \( \varepsilon_2 \) between the cylinders, and let the cylinder radius be \( a \).

**Problem 1**

Let us first consider the case with the E field pointing in the \( z \) direction along the cylinders, \( i.e., \) transverse magnetic (TM) polarization. In the metamaterial (low–frequency) limit, the E field is then approximately constant inside a unit cell of the photonic crystal. The effective relative permittivity \( \varepsilon_{zz} \) of the metamaterial is defined as the mean of the D field over the unit cell divided by the mean of the E field times \( \varepsilon_0 \) over the unit cell. Show that for a \( z \)-polarized field,

\[
\varepsilon_{zz} = \varepsilon_2 + (\varepsilon_1 - \varepsilon_2) f, \tag{1}
\]

where the fill factor \( f \) is the area of the cylinder relative to the area of the unit cell,

\[
f = \pi a^2/A_u = \pi a^2/(bh). \tag{2}
\]

The area \( A_u \) of the unit cell is the base line \( b \) times the height \( h \).

**Problem 2**

Let us then consider a TE-polarized field, with the E field lying in the \( x-y \) plane, perpendicular to the cylinders, again in the metamaterial limit. Limiting our analysis to a small fill factor, we may consider the E field to be approximately constant inside and between the cylinders. There is then a region near the outside of each cylinder where the field is not constant, and where we may use the low-frequency approximation that the E field is the gradient of a potential \( V(r, \varphi) \) that is continuous everywhere, and has the form

\[
V_1(r, \varphi) = -E_0 \frac{2\varepsilon_2}{\varepsilon_2 + \varepsilon_1} r \cos \varphi = -E_0 \frac{2\varepsilon_2}{\varepsilon_2 + \varepsilon_1} x \quad \text{for } r < a \quad \text{(inside the cylinder)} \tag{3}
\]

\[
V_2(r, \varphi) = -E_0 \left( r + \frac{\varepsilon_2 - \varepsilon_1 a^2}{\varepsilon_2 + \varepsilon_1} r \right) \cos \varphi = -E_0 \left( x + \frac{\varepsilon_2 - \varepsilon_1 a^2 x}{\varepsilon_2 + \varepsilon_1} \right) \cos \varphi \tag{4}
\]

\[
= -E_0 \left( x + \frac{\varepsilon_2 - \varepsilon_1 a^2 x}{\varepsilon_2 + \varepsilon_1} \right) \cos \varphi \quad \text{for } r > a \quad \text{(outside the cylinder)} \tag{5}
\]

Show that the potential (3) inside the cylinder yields a constant E field that points in the \( x \) direction and is equal to

\[
E_{x,1} = \frac{2\varepsilon_2}{\varepsilon_2 + \varepsilon_1} E_0. \tag{6}
\]

**Problem 3**

From (5), derive expressions for the \( x \) and \( y \) components of the E field outside the cylinder with radius \( a \). Show that the mean of the E field points in the \( x \) direction, when the mean is taken over the cross-sectional area outside the cylinder of radius \( a \) and inside the rectangular unit cell. Show that this mean is equal to \( E_0 \), regardless of the size of the unit cell.
Problem 4

Show that when the cylinders are far from each other, we get the following approximations for the means of $E_x$ and $D_x$ over the unit cell,

\[
\begin{align*}
\bar{E}_x & = \left( 1 + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} f \right) E_0, \\
\bar{D}_x & = \varepsilon_2 \varepsilon_0 \left( 1 - \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} f \right) E_0,
\end{align*}
\]

resulting in the effective relative permittivity

\[
\varepsilon_{xx} \approx \varepsilon_2 \left( 1 - \frac{2(\varepsilon_2 - \varepsilon_1)}{\varepsilon_2 + \varepsilon_1} f \right).
\]

Problem 5

Now let us consider the general case with cylinders that are not far from each other, but restrict ourselves to a rectangular unit cell with width $b$ and height $h$. We note that if the E field is $x$-polarized in the center of the cylinder in a rectangular unit cell, the E field is purely $x$ polarized in all the mirror planes of the structure, $x$-z planes and $y$-z planes going through the centers of the cylinders and in the middle between cylinders. We note that everywhere inside the unit cell,

\[
r < d = \frac{1}{2} \sqrt{b^2 + h^2}.
\]

Instead of a single cosine contribution like in (3), we then need a sum of cosine terms, a so-called multipole expansion, to represent the E field, both inside and outside the cylinder. Inside the cylinder (for $r < a$) we may use the following expressions for the $x$ and $y$ components of the E field

\[
\begin{align*}
E_{x,1}(r, \varphi) & = \sum_{m=0}^{M-1} E_m \frac{2\varepsilon_2}{\varepsilon_2 + \varepsilon_1} \frac{r^{2m}}{d^{2m}} \cos (2m\varphi), \\
E_{y,1}(r, \varphi) & = -\sum_{m=0}^{M-1} E_m \frac{2\varepsilon_2}{\varepsilon_2 + \varepsilon_1} \frac{r^{2m}}{d^{2m}} \sin (2m\varphi).
\end{align*}
\]

The corresponding expressions for the E field outside of the cylinders (for $r > a$) are

\[
\begin{align*}
E_{x,2}(r, \varphi) & = \sum_{m=0}^{M-1} E_m \left( \frac{r^{2m}}{d^{2m}} \cos (2m\varphi) - \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} \frac{a^{4m+2}}{d^{2m}r^{2m+2}} \cos (2m\varphi + 2\varphi) \right), \\
E_{y,2}(r, \varphi) & = -\sum_{m=0}^{M-1} E_m \left( \frac{r^{2m}}{d^{2m}} \sin (2m\varphi) - \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} \frac{a^{4m+2}}{d^{2m}r^{2m+2}} \sin (2m\varphi + 2\varphi) \right).
\end{align*}
\]

We note that for an $x$-polarized field in a rectangular unit cell, only terms with even multiples $2m$ of the angle $\varphi$ are needed in the multipole expansions (11)-(14).

Show that with $E_x$ and $E_y$ given by the multipole expansions (11)-(14), the average of $E_y$ over a rectangular unit cell is zero.

Problem 6 (Matlab)

We can find the expansion coefficients $E_m$ in the series (11)-(14) via point matching. So let us require $E_y(r, \varphi)$ in (14) to be minimized in $2M - 1$ different positions around the unit cell, given by $2M - 1$ different values for the angle $\varphi$

\[
\varphi_p = \frac{\pi}{4M} p, \quad p = 1, 2, \ldots, (2M - 1).
\]
The corresponding distances from the origin are

\[ r_p = \frac{b}{2 \cos \varphi_p} \quad \text{if} \quad \tan \varphi_p < h/b \quad \text{and} \quad r_p = \frac{h}{2 \sin \varphi_p} \quad \text{if} \quad \tan \varphi_p > h/b. \quad (16) \]

Use the \(2M-1\) equations obtained by setting \(E_y(r_p, \varphi_p)/E_0 = 0\) in (14) for \(p = 1, 2, \ldots (2M-1)\) to set up an overdetermined set of linear equations in Matlab and determine \(E_m/E_0\) for \(m = 1, 2, \ldots (M-1)\). Then compute the field in the middle between the cylinders, \(E_x(r = b/2, \varphi = 0)/E_0\). Do the calculation for \(\varepsilon_1 = 2, \varepsilon_2 = 1, \) and \(M = 10\) terms in the series expansion, for two cases of a rectangular unit cell, a tall cell with \(b = 3a\) and \(h = 4a\), and a wide cell with \(b = 4a\) and \(h = 3a\).

Hint: An overdetermined system of linear equations can be solved in Matlab with the help of the matrix divide operation.

Finally, do a numerical average over the unit cell to obtain \(\overline{E}_x\) and \(\overline{D}_x\) for both the tall and the wide unit cells, and compare the numerically computed averages with the formulas (7) and (8).

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