MAT-IN3110, Autumn 2017
Compulsory Assignment 2

Deadline 9 November, 14:30

Assignments should be submitted through the Devilry system.

1 Polynomial interpolation

Let $p(x)$ be the unique polynomial of degree $\leq n$ that interpolates the values $f_i = f(x_i)$, $i = 0, 1, \ldots, n$, of a function $f : \mathbb{R} \to \mathbb{R}$ at the distinct points $x_0, x_1, \ldots, x_n \in \mathbb{R}$.
(a) Write down the Lagrange form of p.
(b) Consider the uniform and Chebyshev points, respectively,

$$x_i = -1 + \frac{2i}{n}, \qquad x_i = \cos\left(\frac{2i+1}{n+1}\frac{\pi}{2}\right), \qquad i = 0, 1, \ldots, n.$$

For each degree $n = 2, 4, 6, \ldots, 16$, find the interpolant p to the function $f(x) = 1/(1+25x^2)$ for each of the two point sets. Is p a good approximation to f? Plot the two polynomials p in the case $n = 16$.
(c) For each degree $n = 2, 4, 6, \ldots, 16$, choose random values $f_0, f_1, \ldots, f_n$ in the interval $[0, 1]$ and find the interpolant p for each of the two point sets. What do you observe? Plot the two polynomials p in the case $n = 16$.
(d) From the Lagrange form, derive the so-called *barycentric form* of p, i.e., express p in the form

$$p(x) = \sum_{i=0}^{n} \frac{w_i f(x_i)}{x - x_i} \Bigg/ \sum_{i=0}^{n} \frac{w_i}{x - x_i}$$

for weights $w_0, w_1, \ldots, w_n$. What are these weights when the points x_i are uniformly spaced?

2 Orthogonal polynomials

Recall that the Chebyshev polynomials are defined by

$$T_n(x) = \cos(n \arccos(x)), \qquad n \geq 0.$$

So, $T_0(x) = 1$ and $T_1(x) = x$.
(a) Derive the three-term recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad n \geq 1.$$

(b) Prove that the Chebyshev polynomials are orthogonal on $[-1, 1]$ with respect to the weight function $w(x) = (1 - x^2)^{-1/2}$, i.e., show that if $m \neq n$ then $\langle T_m, T_n \rangle = 0$ where

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)(1 - x^2)^{-1/2} \, dx.$$

3 Bernstein polynomials

What are the Bernstein polynomials of degree n on the interval $[0, 1]$? What important properties do they have that make them useful for designing curves (Bezier curves)?