Mid-term exam in MAT-INF 1100, October 17, 2003
Time: 9.00–11.00

Candidate number:

The first 15 problems count two points each while the last 5 count 4 points each. The maximum score is therefore 50 points. There are 5 alternative answers for each problem, but only one is correct. If your answer is wrong or you do not tick off one of the answers you get zero points. In other words, you will not be penalised with negative points for answering incorrectly. *Good luck!*

Problem and answer sheets

1) The binary number 1100101 is the same as the decimal number

- ☐ 50
- ☐ 104
- ☐ 101
- ☐ 93
- ☐ 81

2) Written in binary the number 140 becomes

- ☐ 10110100
- ☐ −1010100
- ☐ 10001100
- ☐ 10110001
- ☐ 11001100

3) The real number $1/(1+\sqrt{2})$ is

- ☐ $1-\sqrt{2}$
- ☐ a rational number
- ☐ a natural number
- ☐ not defined
- ☐ an irrational number

4) The real number $\dfrac{4}{3\sqrt{5}-5} - \dfrac{3}{\sqrt{5}}$ is

- ☐ an irrational number
- ☐ a negative number
- ☐ 5
- ☐ 0
- ☐ a rational number

5) The greatest lower bound of the set $\{x : |2x+1| < 1\}$ is

☐ 0
☐ 2
☐ $\sqrt{2}$
☐ -1
☐ -2

6) The least upper bound of the set $\{x : x^2 - 2 < 2x\}$ is

☐ $\sqrt{3}$
☐ $1-\sqrt{3}$
☐ 2
☐ $1+\sqrt{3}$
☐ 1

7) Suppose that we multiply together the factors in the expression $(a+1)^{31}$, where a is different from 0, what will the coefficient multiplying a^{29} be?

☐ 17
☐ 359
☐ 465
☐ 431
☐ 546

8) Which of the following expressions may give a large relative error for certain values of a and b when the computations are performed with floating-point numbers, we ignore underflow and overflow, and a and b are such that the computations make sense?

☐ $\sqrt{a-b}$
☐ a^2b
☐ $2a$
☐ a/b
☐ $\sqrt{ab}$

9) Which one of the following statements are true?

☐ The natural numbers are not a subset of the rational numbers
☐ When the square root of a positive, integer number is not an integer it is an irrational number
☐ Round-off errors always cause problems when working with integers
☐ There are only finitely many rational numbers
☐ There are infinitely many 64-bit integers

10) The function f is defined on the interval $[a, b]$, is continuous, and satisfies the condition $f(a) \cdot f(b) < 0$. We apply the bisection method to determine an approximation to a zero in I, and after 11 iterations we know that the absolute error lies in the interval $(0.0014, 0.0015)$. Then the interval I must have length $b - a$ equal to

- ☐ 3
- ☐ 1
- ☐ 2
- ☐ 0.5
- ☐ e

11) Which one of the following difference equations is linear?

- ☐ $x_n - \log(x_{n-1}) + x_{n-2} = 0$
- ☐ $e^{\sin x_n} + x_{n-1} = 0$
- ☐ $x_n^2 + x_{n-2} = 0$
- ☐ $x_n + n^{1/2}\, x_{n-1} + x_{n-2} = 0$
- ☐ $\sqrt{x_n} - x_{n-1} = 0$

12) The solution of the difference equation

$$x_{n+2} - x_{n+1} - 2x_n = 0, \qquad x_0 = 0, \quad x_1 = -3$$

is given by

- ☐ $x_n = (1 - 4^n)/2$
- ☐ $x_n = (-1)^n - 2^n$
- ☐ $x_n = -3n$
- ☐ $x_n = n - 1 - 3^n$
- ☐ $x_n = -(n-1)^2 - 2^n$

13) A difference equation has characteristic equation with roots $r_1 = 3 + 2i$ and $r_2 = 3 - 2i$. The difference equation is then given by

- ☐ $x_{n+2} - 2x_{n+1} + 2x_n = 0$
- ☐ $x_{n+2} - x_n = 0$
- ☐ $x_{n+2} + x_{n+1} - x_n = 0$
- ☐ $x_{n+2} - 6x_{n+1} + 13x_n = 0$
- ☐ $x_{n+2} - 8x_{n+1} + x_n = 0$

14) The solution of the difference equation

$$x_{n+2} + 2x_{n+1} + 4x_n = 0, \qquad x_0 = 0, \quad x_1 = \sqrt{3}$$

is given by

- ☐ $x_n = 4^n - 1$
- ☐ $x_n = 2^n \cos(2n\pi/3)$
- ☐ $x_n = 2^n \sin(2n\pi/3)$
- ☐ $x_n = n\sqrt{3}$
- ☐ $x_n = 2^n \sin(n\pi/3)$

15) Numerical simulation of the difference equation $x_{n+2} - \frac{7}{3}x_{n+1} + \frac{2}{3}x_n = 0$ with 64-bit floating-point numbers and initial values $x_0 = 1$ og $x_1 = 1/3$ will result in

- ☐ major problems with round-off errors
- ☐ no problems with round-off errors
- ☐ $x_n = (n+1)^2/3$
- ☐ $x_n = 0$
- ☐ $x_n = \texttt{NaN}$

16) We let P_n denote the statement that the formula

$$\sum_{i=1}^{n} 2i - 1 = (n-1)^2 + 1$$

is true for $n \geq 1$. To prove this by induction we may proceed as follows:

1. When $n = 1$ both sides of the formula are 1, so the formula is correct in this case.
2. Suppose we have proved that $P_1, \ldots, P_k$ is true, to complete the proof we must prove that then P_{k+1} is also true. Since P_k is true we have

$$\begin{aligned}\sum_{i=1}^{k+1}(2i-1) = \sum_{i=1}^{k}(2i-1) + 2k + 1 &= (k-1)^2 + 1 + 2k + 1 \\ &= k^2 - 2k - 1 + 1 + 2k + 1 = k^2 + 1.\end{aligned}$$

In other words, if P_k is true, then P_{k+1} must also be true and therefore P_n is true for all $n \geq 1$.

Which one of the following statements are true?

- ☐ The statement P_n is true, but part 2 of the proof is wrong
- ☐ The statement P_n is wrong and part 2 of the proof is wrong
- ☐ The statement P_n is wrong, and both part 1 and part 2 of the proof is wrong
- ☐ Both the statement P_n and the proof are correct
- ☐ The proof is correct, but it is not a proof by induction

17) Let f_n denote the Fibonacci sequence defined by

$$f_n = f_{n-1} + f_{n-2}, \qquad n \geq 0, \quad f_0 = 0, \quad f_1 = 1. \tag{1}$$

In this problem we are going to find a formula for the solution of the difference equation

$$x_n = x_{n-1} \cdot x_{n-2}, \qquad n \geq 0, \quad x_0 = 1, \quad x_1 = 2.$$

We believe that the following statement is true:

P_n: For all integers $n \geq 0$ it holds that $x_n = 2^{f_n}$, where f_n is the Fibonacci sequence given by (1) above.

We try to prove this by induction:

1. We observe straight away that $x_0 = 1 = 2^0 = 2^{f_0}$ og $x_1 = 2 = 2^1 = 2^{f_1}$, so P_0 and P_1 are both true.

2. Suppose we have shown that P_n is true for $n = 0, \ldots, k$, we must show that then P_{k+1} is also true. We have

$$x_{k+1} = x_k \cdot x_{k-1} = 2^{f_k} 2^{f_{k-1}} = 2^{f_k + f_{k-1}} = 2^{f_{k+1}}$$

where the last equality follows from (1). From this we conclude that P_n must be true for all $n \geq 0$.

Which one of the following statements are true?

- ☐ The statement P_n is true, but part 1 of the proof is wrong
- ☐ The statement P_n is true, but part 2 of the proof is wrong
- ☐ The statement P_n is wrong, but part 2 of the proof is correct
- ☐ Both the statement P_n and the proof are correct
- ☐ The statement P_n is wrong, but part 1 of the proof is correct

18) The solution of the inhomogenous difference equation

$$x_{n+2} - 4x_{n+1} + 4x_n = 3^n$$

where $x_0 = 1$ og $x_1 = 1$ is given by

- ☐ $x_n = 2^n - n$
- ☐ $x_n = 2^n - n\,(-3)^n$
- ☐ $x_n = (n^2 + n)/2$
- ☐ $x_n = n + 2 - 2^n$
- ☐ $x_n = 3^n - n2^n$

19) The solution of the inhomogenous difference equation

$$x_{n+1} - 2x_n = n^2$$

where $x_0 = 0$ is given by

- ☐ $x_n = n$
- ☐ $x_n = n^2$
- ☐ $x_n = 3(2^n - 1) - 2n - n^2$
- ☐ $x_n = 2^n - 1$
- ☐ $x_n = 2^n - n^2 - 1$

20) We apply Newton's method $x_{n+1} = x_n - f(x_n)/f'(x_n)$ to the function $f(x) = x^2 - A$ where A is a positive, real number. If we denote the error by $e_n = x_n - \sqrt{A}$, we have

- ☐ $e_{n+1} = \dfrac{e_n}{2x_n}$
- ☐ $e_{n+1} = \dfrac{e_n^2}{2x_n}$
- ☐ $e_{n+1} = \dfrac{e_n^2}{x_n^2}$
- ☐ $e_{n+1} = \dfrac{e_n e_{n-1}}{x_n}$
- ☐ $e_{n+1} = \log e_n$

That's it!!