October 13, 2008

MAT-INF 1100: Compulsory Assignment 2

Deadline: October 31, 2008, 14:30

Information

The written answers are to be handed in at the office of the Maths Department on the 6th floor in Niels Henrik Abel's building (7. etasje in Norwegian) by 14:30 on Friday, September 31. The answers should be written by yourself (by hand or computer). The program in problem 2d should be a computer printout. Please make sure that the answers are written clearly and in the correct order so that the teacher who does the marking can easily understand and find everything.

Students who become ill or for other reasons wish to apply for postponement or exemption from this assignment should contact the student administration in the Maths Department (email: studieinfo@math.uio.no, telephone 22 85 58 88) in good time before the deadline.

Students are encouraged to collaborate on this assignment, and the teaching assistants will answer general questions, but cannot provide complete solutions. *The final answers that you hand in must be produced by yourself, and you must be able to explain the contents of your answers if you are called in for an oral examination (may happen if there is suspicion of copying).*

Remember that both of the compulsory assignments in MAT-INF 1100 must be passed before you are allowed to sit the final course exam. *To pass this second assignment you must make serious attempts at solving all the problems, and at least six of the ten problems should be correctly answered.*

If you have collaborated with someone, please write their name here:

Problems

Problem 1. By using some numerical method we have solved a differential equation which gives the speed v of an object falling towards the earth. The result is a sequence of approximations $(t_i, v_i)_{i=0}^N$ to points on the solution, where $t_i = ih$ for a suitable value h.

a) Give a method (you may for example write it in the same form as the algorithms in the lecture notes) to compute an approximation to the object's accelleration $a(t) = v'(t)$, based on the computed values (t_i, v_i) of the speed.

b) Give a method for computing an approximation to the object's hegiht above sea level $y(t)$ from the computed values when $v(t) = y'(t)$ og $y(0) = a$.

Problem 2. In this problem you are going to make the Trapezoid method more effective. Suppose you are going to perform numerical intergration with the Trapezoid method on the interval $[a, b]$ by computing a sequence of approximations I_0, I_1, I_2, ... To compute the approximation I_n, the interval $[a, b]$ is split into 2^n subintervals of equal width, the function f is approximated by the secant on each subinterval, and the integral of f is approximated by the integral of the secant.

a) Show that this gives

$$I_n = h\left(\frac{f(a) + f(b)}{2} + \sum_{i=1}^{2^n-1} f(a + ih)\right), \tag{1}$$

the so-called Trapezoid rule.

b) Some of the function values involved in the computation of I_n also occur in the computation of I_{n-1}. Explain which these values are.

c) Let h be the step length for the computation of I_n. Explain why

$$I_n = \frac{I_{n-1}}{2} + h\sum_{i=1}^{2^{n-1}} f\big(a + (2i-1)h\big). \tag{2}$$

How can this formula be exploited to avoid computing function values several times when we compute I_1, I_2, ...?

d) On the file `trapes.py` you will find a Python-program that computes an approximation to $\int_0^1 \cos x \, dx$ with relative error less than 10^{-12}, based on the formula (1). Rewrite the program so that you exploit the formula (2). How much faster is the new program?

Problem 3. We have the differential equation

$$x' - x^2 = 1, \quad x(0) = 1. \tag{3}$$

a) Find the solution $x(t)$ of the equation analytically. (Hint: The equation is separable.)

b) Solve the equation numerically on the interval $[0, 0.6]$ by taking 6 steps with Euler's method (with calculator or computer). Plot the numerical solution together with the exact solution (by hand or by computer).

c) Repeat (b), by use Euler's midpoint method instead of Euler's method. Plot the resulting numerical solution in the plot you produced in (b).

d) An alternative to the two methods above is the following. Suppose we are gong to solve $x' = f(t, x)$. The new method is based on letting the step from the approximation (t_k, x_k) to the approximation (t_{k+1}, x_{k+1}) be computed by

$$x_{k+1} = x_k + hf(t_k + h/2, (x_{k+1} + x_k)/2).$$

For the equation (3) this gives

$$x_{k+1} = x_k + h\left(1 + \frac{(x_k + x_{k+1})^2}{4}\right). \tag{4}$$

This equation can be solved with respect to x_{k+1}, which yields

$$x_{k+1} = \frac{2 - hx_k - 2\sqrt{1 - h^2 - 2x_k h}}{h}. \tag{5}$$

Derive a method which computes x_{k+1} using this formula. Are there any limitations on which h can be used?

Perform 6 steps as for the other methods, and add this solution to the plot of the other solutions. Which method seems to work best?

e) (May be skipped.) The equation (4) has two solutions, and in (5) we select one of these. Explain why we do not use the other solution.

Good luck!!